

NORMALIZED BERKOVICH SPACES AND SURFACE SINGULARITIES

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ABSTRACT. We define normalized versions of Berkovich spaces over a trivially valued field k , obtained as quotients by the action of $\mathbb{R}_{>0}$ defined by rescaling semivaluations. We associate such a normalized space to any special formal k -scheme and prove an analogue of Raynaud's theorem, characterizing categorically the spaces obtained in this way. This construction yields a locally ringed G -topological space, which we prove to be locally isomorphic to a Berkovich space over the field $k((t))$ with a t -adic valuation. These spaces can be interpreted as non-archimedean models for the links of the singularities of k -varieties, and allow to study the birational geometry of k -varieties using techniques of non-archimedean geometry available only when working over a field with nontrivial valuation. In particular, we prove that the structure of the normalized non-archimedean links of surface singularities over an algebraically closed field k is analogous to the structure of non-archimedean analytic curves over $k((t))$, and deduce a characterization of the log essential valuations, i.e. those valuations whose center on every log resolution of the given surface is a divisor.

1. INTRODUCTION

Berkovich's geometry is an approach to non-archimedean analytic geometry developed in the late nineteen-eighties and early nineteen-nineties by Berkovich in [Ber90] and [Ber93]. To overcome the problems given by the fact that the metric topology of any valued field is totally disconnected, Berkovich adds many points to the usual points of a variety X (not unlike what happens in algebraic geometry with generic points), to obtain an *analytic space* X^{an} , which is a locally ringed space with very nice topological properties and whose points can be seen as *real semivaluations*.

One important feature of Berkovich's theory is that it works also over a trivially valued base field, for example \mathbb{C} . This gives rise to objects which are far from being trivial, resembling some spaces studied in valuation theory, but carrying in addition the analytic structure, and containing a lot of information about the singularities of X . For example, Thuillier [Thu07] obtained the following result (generalizing a theorem by Stepanov): if X is a variety over a perfect field k , then the homotopy type of the dual intersection complex of the exceptional divisor of a log resolution of X does not depend on the choice of the log resolution. To prove this, he associates to a subvariety Z of a k -variety

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X a k -analytic space that can be called the *punctured tubular neighborhood* of Z^{an} in X^{an} . It is a subspace of X^{an} , invariant under modifications of the pair (X, Z) , consisting of all the semivaluations on X which have center on Z but are not semivaluations on Z .

In this paper we define a normalized version $T_{X,Z}$ of this punctured tubular neighborhood, by taking the quotient of the latter by the group action of $\mathbb{R}_{>0}$ which corresponds to rescaling semivaluations. The space $T_{X,Z}$ can be thought of as a *non-archimedean model* of the *link* of Z in X . It is a locally ringed space in k -algebras, endowed with the pushforward of the Grothendieck topology and structure sheaf from the punctured tubular neighborhood, and it can be seen as a wide generalization of Favre and Jonsson's *valuative tree*, an object which has important applications to the dynamics of complex polynomials in two variables. Indeed, the valuative tree is homeomorphic to the topological space underlying $T_{\mathbb{A}_{\mathbb{C}}^2, \{0\}}$, but the latter has much more structure. Moreover, $T_{X,Z}$ can also be thought of as a compactification of the normalized valuation space considered in [JM12] and [BdFFU13], as the latter is homeomorphic to the subset of $T_{X,Z}$ consisting of all the points which are actual valuations on the function field of X . More generally, we associate a *normalized space* $T_{\mathcal{X}}$ to any *special formal k -scheme* \mathcal{X} . If X is a k -variety and Z is a closed subvariety of X , then the formal completion $\widehat{X/Z}$ of X along Z is a special formal k -scheme, and we have $T_{\widehat{X/Z}} \cong T_{X,Z}$.

The crucial property of $T_{\mathcal{X}}$ is the following: while not an analytic space itself, as a locally ringed G -topological space in k -algebras the normalized space $T_{\mathcal{X}}$ is locally isomorphic to an analytic space over the field $k((t))$ with a t -adic absolute value. Attention should be paid to the fact that these local isomorphisms are not canonical, and in general they do not induce a global $k((t))$ -analytic structure on $T_{\mathcal{X}}$. In particular, this result explains why the valuative tree looks so much like a Berkovich curve defined over $\mathbb{C}((t))$. This interpretation permits to study $T_{\mathcal{X}}$, and thus deduce information about \mathcal{X} , with many tools of non-archimedean analytic geometry, including the ones that work only over non-trivially valued fields. We have only recently learned about the article [BBT13]. There the authors use the punctured tubular neighborhood of Z^{an} in X^{an} to encode the descent data necessary to glue a coherent sheaf on a formal neighborhood of Z to a coherent sheaf on $X \setminus Z$. The result we just described on the structure of $T_{\mathcal{X}}$ is conceptually similar to the results of Sections 4.2, 4.3, 4.4 and 4.6 of *loc. cit.*

We define an affinoid domain of $T_{\mathcal{X}}$ as a G -open subspace V of $T_{\mathcal{X}}$ which is isomorphic to a strict $k((t))$ -affinoid space, and we show that this definition does not depend on the choice of a $k((t))$ -analytic structure on V . This is done by showing, following [Liu90], that a $k((t))$ -analytic space is strictly affinoid if and only if it is Stein, compact and its ring of analytic functions bounded by one is a special k -algebra. Every normalized space $T_{\mathcal{X}}$ is compact, G -covered by finitely many affinoid domains, and this allows us to characterize the category of all the locally ringed G -topological spaces of the form $T_{\mathcal{X}}$ as

the localization of the category of special formal k -schemes by the class of admissible formal blowups. This is a “normalized spaces version” of a classical theorem of Raynaud for non-archimedean analytic spaces (see [BL85, 4.1]).

We then apply normalized spaces to the study of surface singularities. While the importance of valuations in the study of resolutions of surface singularities was emphasized already in the work of Zariski and Abhyankar (see [Zar39] and [Abh56]), in our work also the additional structure given by the sheaf of analytic functions plays an important role. If k is an algebraically closed field, X is a k -surface and Z is a subspace of X containing its singular locus, by the structure theorem discussed above the normalized space $T_{X,Z}$ behaves like a non-archimedean analytic curve over $k((t))$. The theory of such curves is well understood, thanks to work of Bosch and Lütkebohmert [BL85] ([Ber90, Chapter 4] for Berkovich spaces). In particular, there is a correspondence between (semistable) models and (semistable) vertex sets (see [Duc], [Tem10, Chapter 6] and [BPR14]). We prove an analogue of this result for the normalized space $T_{X,Z}$. After showing how to construct formal log modifications of the pair (X, Z) with prescribed exceptional divisors, we characterize among those modifications the ones which correspond to a log resolution of (X, Z) by performing a careful study (analogous to [BL85, 2.2 and 2.3] and [Ber90, 4.3.1]) of the fibers of the map sending every semivaluation to its center on the modification. Our main source of inspiration in developing this approach was Ducros’s work [Duc].

The strategy described above leads to a characterization in terms of the local structure of $T_{X,Z}$ of the *log essential valuations* on (X, Z) , i.e. those valuations whose center on every log resolution of (X, Z) is a divisor. Whenever k is the field of complex numbers and Z is the singular locus of X , this is related to a famous conjecture of Nash from the nineteen-seventies (but published only in 1995 in [Nas95]) involving the *arc space* X_∞ of a complex variety X . Nash constructed an injective map from the set of irreducible components of the subspace of X_∞ consisting of the arcs centered in the singular locus of X to the set of *essential valuations* on X , i.e. the valuations whose center on every resolution of X is a divisor, and asked if this map is surjective. While this is known to be false if $\dim(X) \geq 3$ (see [dF13]), for complex surfaces a proof was given by Fernández de Bobadilla and Pe Pereira in [FdBP12]. More recently de Fernex and Docampo [dFD14] proved that in arbitrary dimension every valuation which is terminal with respect to the minimal model program over X is in the image of the Nash map, deducing a new proof of de Bobadilla–Pereira’s theorem. The class of log essential valuations can be larger than the set of Nash’s essential valuations, since in some cases the exceptional locus of the minimal resolution of X may not be a divisor with normal crossings. However, for many classes of singularities (e.g. rational singularities) these two notions coincide.

We now give a short overview of the content of the paper. In Section 2 we recall some basic constructions of the theories of formal schemes and Berkovich spaces. In Section 3 we define the normalized space of a special

formal k -scheme, while in Section 4 we prove the structure theorem of normalized spaces and deduce some interesting consequences. In Section 5 we define affinoid domains in a normalized space, and show that this notion is independent on the choice of a $k((t))$ -analytic structure. This leads to the normalized version of Raynaud's theorem in Section 6. We then move to the study of pairs (X, Z) , where X is a k -surface and Z is a closed subvariety of X containing its singular locus. Section 7 contains the correspondence theorem between formal modifications of (X, Z) and vertex sets. In Section 8 we study discs and annuli in $T_{X,Z}$; they are used in Section 9, where we describe the formal fibers of the specialization map. In Section 10 we show how these techniques lead to the characterization of log essential valuations.

Several examples have been given for the reader who might want to quickly reach a basic understanding of the applications of normalized spaces to the study of surface singularities, without spending much time learning formal and Berkovich geometry. This reader should pay attention to the examples 2.12, 2.23, 3.14, 4.12, and might benefit from reading the short note [Fan14], where some of the results of this paper were announced.

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2. SPECIAL FORMAL SCHEMES AND THEIR BERKOVICH SPACES

In the section we recall the notions of special formal schemes and the associated Berkovich spaces. For a detailed study of noetherian formal schemes we refer the reader to [Ill05] or [Bos14]; a quick introduction is [Nic08]. Special formal schemes are treated for example in [dJ95] and [Ber96a].

(2.1) Let R be a complete discrete valuation ring, K the fraction field of R and k its residue field. By definition we allow R to be a trivially valued field k . A *formal R -scheme* is a noetherian formal scheme endowed with a (not necessarily adic) morphism of noetherian formal schemes to $\mathrm{Spf} R$.

(2.2) A topological R -algebra A is said to be a *special R -algebra* if it is a noetherian adic ring and the quotient A/J is a finitely generated R -algebra for some ideal of definition J of A . A formal R -scheme \mathcal{X} is said to be a *special formal R -scheme* if it is separated and locally isomorphic to the formal spectrum of a special R -algebra. In particular the reduction \mathcal{X}_0 of \mathcal{X} is a reduced and separated scheme locally of finite type over k .

(2.3) By [Ber96a, 1.2], special R -algebras are the adic R -algebras of the form

$$R\{X_1, \dots, X_n\}[[Y_1, \dots, Y_m]]/I \cong R[[Y_1, \dots, Y_m]]\{X_1, \dots, X_n\}/I,$$

with ideal of definition generated by an ideal of definition of R and by the Y_i 's. Recall that if A is a I -adic topological ring, then $A\{X_1, \dots, X_n\} := \varprojlim_{n \geq 1} (A/I^n)[X_1, \dots, X_n]$ is the algebra of converging power series over A in the variables (X_1, \dots, X_n) . Since every R -algebra topologically of finite type is special (we can take $m = 0$ above), every formal R -scheme of finite type is a special formal R -scheme. On the other hand, a special formal R -scheme is of finite type if and only if its structure morphism to $\mathrm{Spf}(R)$ is adic.

(2.4) Example: the algebraic case. If X is a separated R -scheme locally of finite type and Z is a subscheme of the special fiber $X \otimes_R k$ of X , then the formal completion $\mathcal{X} = \widehat{X/Z}$ of X along Z is a special formal R -scheme. In this case, $\mathcal{X}_0 = Z_{\mathrm{red}}$. For example, if $X = \mathbb{A}_R^2 = \mathrm{Spec}(R[X_1, X_2])$ and Z is the origin of the special fiber of X , then $\widehat{X/Z} \cong \mathrm{Spf}(R[[X_1, X_2]])$.

(2.5) A special formal R -scheme \mathcal{X} is said to be *normal* if it can be covered by affine subschemes $\mathrm{Spf}(A)$ with A normal. Since the rings A are excellent, this is equivalent to the normality of all completed local rings of \mathcal{X} .

(2.6) If A is a special k -algebra and t is an element of its largest ideal of definition, then t is topologically nilpotent and therefore it induces a morphism $k[[t]] \rightarrow A$ which canonically makes A into a special $k[[t]]$ -algebra. Conversely, any special $k[[t]]$ -algebra is canonically a special k -algebra. We will sometimes denote a special formal $k[[t]]$ -scheme by \mathcal{X}_t ; \mathcal{X} will then denote \mathcal{X}_t seen as a special formal k -scheme.

(2.7) Let \mathcal{X} be a noetherian formal scheme with largest ideal of definition \mathcal{J} and let \mathcal{I} be a coherent ideal sheaf on \mathcal{X} . The *formal blowup of \mathcal{X} at \mathcal{I}* is the morphism of formal schemes

$$\mathcal{X}' := \varinjlim_{n \geq 1} \mathrm{Proj} \left(\bigoplus_{d=0}^{\infty} \mathcal{I}^d \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{O}_{\mathcal{X}}/\mathcal{J}^n) \right) \longrightarrow \mathcal{X}.$$

We call the closed formal subscheme of \mathcal{X} defined by \mathcal{I} the *center* of the blowup. The formal blowup $\mathcal{X}' \rightarrow \mathcal{X}$ of \mathcal{X} at \mathcal{I} is characterized by the following universal property (see [Bos14, 2.6.9]): \mathcal{X}' is a noetherian formal scheme such that the ideal $f^{-1}\mathcal{I}\mathcal{O}_{\mathcal{X}'}$ is invertible on \mathcal{X}' , and every morphism of noetherian formal schemes $\mathcal{Y} \rightarrow \mathcal{X}$ such that $f^{-1}\mathcal{I}\mathcal{O}_{\mathcal{Y}}$ is invertible on \mathcal{Y} factors uniquely through a morphism of noetherian formal schemes $\mathcal{Y} \rightarrow \mathcal{X}'$. We say that the blowup $\mathcal{X}' \rightarrow \mathcal{X}$ of \mathcal{X} at \mathcal{I} is *admissible* if the ideal \mathcal{I} is \mathcal{J} -open, i.e. contains a power of \mathcal{J} .

(2.8) Example. Let X be a noetherian scheme, let Z be a closed subscheme of X defined by a coherent ideal sheaf \mathcal{J} and denote by $\mathcal{X} = \widehat{X/Z}$ the formal completion of X along Z . Let \mathcal{I} be a \mathcal{J} -open coherent ideal sheaf on X , and $\hat{\mathcal{I}}$ the induced ideal sheaf on \mathcal{X} . Then the formal blowup of \mathcal{X} at $\hat{\mathcal{I}}$ is isomorphic to the formal completion of the blowup $\mathrm{Bl}_{\mathcal{I}}(X)$ of X at \mathcal{I} along $f^*\mathcal{J}\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$, where $f : \mathrm{Bl}_{\mathcal{I}}(X) \rightarrow X$ is the blowup. This is [Nic09, 2.16.(5)].

(2.9) Admissible formal blowups share many properties with blowups of ordinary schemes. In particular, the following facts are proved as for schemes:

- (i) a composition of admissible blowups is an admissible blowup ([Bos14, 2.6.11]);
- (ii) two admissible blowups can be dominated by a third one ([Bos14, 2.6.16] and the previous point);
- (iii) an admissible blowup of an open formal subscheme of \mathcal{X} can be extended to an admissible blowup of \mathcal{X} ([Bos14, 2.6.13]).

(2.10) An admissible blowup of a special formal R -scheme is a special formal R -scheme by [Nic09, 2.17]. Similarly, an admissible blowup of a formal R -scheme of finite type is of finite type.

Berkovich's approach to non-archimedean analytic geometry was developed in [Ber90] and [Ber93]; a good introduction to the theory is [Tem10]. Since the general definition of a K -analytic space is quite technical, we will content ourselves with listing some properties of K -analytic spaces and introducing via examples those spaces which appear in the rest of the paper. In particular, we will recall how to associate a K -analytic space \mathcal{X}^\natural to a special formal R -scheme \mathcal{X} and define the specialization map. This construction was introduced for rigid spaces by Berthelot in [Ber96b] (see also [dJ95, §7] for a detailed exposition), while in the context of Berkovich spaces it was studied in [Ber94] and [Ber96a]. If \mathcal{X} is special over a trivially valued field k , we will also study a subspace of \mathcal{X}^\natural , introduced by Thuillier in [Thu07], which behaves more like a generic fiber for \mathcal{X} (see also [BBT13]).

(2.11) A K -analytic space is a locally compact and locally path connected topological space X with the following additional structure:

- (i) For every point x of X , a completed valued field extension $\mathcal{H}(x)$ of K , called the *completed residue field* of X at x .
- (ii) A G -topology on X , whose admissible open subspaces are called *analytic domains* of X .
- (iii) A local G -sheaf in K -algebras \mathcal{O}_X on X , the *sheaf of analytic functions*.

The G -topology is finer than the usual topology of X , i.e. every open subspace of X is an analytic domain. If V is an analytic domain of X , $x \in V$, $f \in \mathcal{O}_X(V)$, then f can be evaluated in x , yielding an element $f(x)$ of $\mathcal{H}(x)$. Therefore, also $|f(x)| \in \mathbb{R}_+$ makes sense. We refer to [Tem10] for the general definition of the category (An_K) .

(2.12) Example. A fundamental example of K -analytic space is the analytification X^{an} of a K -scheme of finite type X . As a topological space,

$$X^{an} = \{(\xi_x, |\cdot|_x) \mid \xi_x \in X, |\cdot|_x \text{ abs. value on } \kappa(\xi_x) \text{ extending the one of } K\},$$

with the weakest topology such that the map $\rho : X^{an} \rightarrow X$ sending a point $x = (\xi_x, |\cdot|_x)$ to ξ_x is continuous, and for each open U of X and each element f of $\mathcal{O}_X(U)$ the induced map $\rho^{-1}(U) \rightarrow \mathbb{R}$ sending x to $|f(x)| = |f|_x$ is

continuous. The field $\mathcal{H}(x)$ is the completion of $\kappa(\xi_x)$ with respect to $|\cdot|_x$. A morphism of K -schemes of finite type $Y \rightarrow X$ induces a map $Y^{an} \rightarrow X^{an}$. The space X^{an} is connected (respectively Hausdorff, compact) if and only if X is connected (respectively separated, proper). Moreover, whenever X is proper then GAGA type theorems hold (see [Ber90, §3.4, §3.5]). The sheaf $\mathcal{O}_{X^{an}}$ can be thought of as a completion of the sheaf \mathcal{O}_X with respect to some seminorm. For example, the analytic functions on an open U of $\mathbb{A}_K^{n,an}$ are the maps $f : U \rightarrow \prod_{x \in U} \mathcal{H}(x)$ which are locally uniform limits of rational functions without poles. More generally, every locally closed subspace of X^{an} can be canonically given the structure of a K -analytic space.

(2.13) An important class of analytic domains consists of *affinoid domains*. Affinoid domains are compact and Hausdorff, and any analytic domain is G -covered by the affinoid domains it contains. An affinoid domain is isomorphic to the affinoid spectrum $\mathcal{M}(\mathcal{A})$ of an affinoid K -algebra \mathcal{A} . Recall that an affinoid K -algebra is a quotient of a Banach K -algebra of the form $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} = \{\sum_{\underline{i} \in \mathbb{N}^n} a_{\underline{i}} \underline{T}^{\underline{i}} \mid a_{\underline{i}} \in K, \lim_{|\underline{i}| \rightarrow \infty} |a_{\underline{i}}| r^{\underline{i}} = 0\}$ (where $r_i > 0$, and the Banach norm is given by $\|\sum a_{\underline{i}} \underline{T}^{\underline{i}}\| = \max |a_{\underline{i}}| r^{\underline{i}}$), and that the affinoid spectrum $\mathcal{M}(\mathcal{A})$ is the set of bounded multiplicative seminorms on \mathcal{A} , with the topology of pointwise convergence. An affinoid domain is said to be strict if we can take all r_i in $|K^*|$ above. If $V \cong \mathcal{M}(\mathcal{A})$ is an affinoid domain of X , then $\mathcal{O}_X(V) \cong \mathcal{A}$.

(2.14) Example. The *analytic affine n -space* $\mathbb{A}_K^{n,an} = \text{Spec}(K[T_1, \dots, T_n])^{an}$ can be written as the union of the closed polydiscs $D^n(\underline{r}) = \{|T_i| \leq r_i \text{ for all } i\}$ of center 0 and polyradius $\underline{r} = (r_1, \dots, r_n) \in (\mathbb{R}_+^*)^n$. The polydisc $D^n(\underline{r})$ is an affinoid domain of affinoid K -algebra $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. Explicit descriptions of the topological spaces underlying $\mathbb{A}_K^{1,an}$, $\mathbb{A}_k^{1,an}$ and $\mathbb{A}_k^{2,an}$ can be found in [Pay13].

We will now see how to associate a K -analytic space to a special formal R -scheme.

(2.15) If \mathcal{X} is the affine special formal R -scheme of the form

$$\mathcal{X} = \text{Spf} \left(\frac{R\{X_1, \dots, X_n\}[[Y_1, \dots, Y_m]]}{(f_1, \dots, f_r)} \right),$$

then the associated Berkovich space is

$$\mathcal{X}^\natural = V(f_1, \dots, f_r) \subset D^n \times_K D_-^m \subset \mathbb{A}_K^{n+m,an},$$

where $D^n = D^n(\underline{1})$ is the n -dimensional closed unit disc in $\mathbb{A}_K^{n,an}$ (as in 2.14), $D_-^m = \{x \in \mathbb{A}_K^{m,an} \mid |T_i(x)| < 1 \text{ for all } i\}$ is the m -dimensional open unit disc in $\mathbb{A}_K^{m,an}$ and $V(f_1, \dots, f_r)$ denotes the zero locus of the f_i . This construction is functorial, sending an open immersion to an embedding of a closed subdomain, therefore it globalizes to general special formal R -schemes by gluing. If \mathcal{X} is of finite type over R , this construction coincides with the one by Raynaud (see [Ray74] or [BL93, §4]) and \mathcal{X}^\natural is compact.

(2.16) Example. If $\mathcal{X} = \mathrm{Spf}(R\{T\})$, then $\mathcal{X}^\triangleright$ is the closed unit disc in $\mathbb{A}_K^{1,an}$. If $\mathcal{X} = \mathrm{Spf}(R[[T]])$, $\mathcal{X}^\triangleright$ is the open unit disc in $\mathbb{A}_K^{1,an}$. Note that if $K = k$ is trivially valued, the latter is homeomorphic to the interval $[0, 1[$.

(2.17) If $\mathcal{X} = \mathrm{Spf}(R\{X_1, \dots, X_n\}[[Y_1, \dots, Y_m]]/(f_1, \dots, f_r))$ is an affine special formal R -scheme, then its associated Berkovich space $\mathcal{X}^\triangleright$ is the increasing union $\mathcal{X}^\triangleright = \bigcup_{0 < \varepsilon < 1} W_\varepsilon$, where W_ε is the subspace of $\mathcal{X}^\triangleright$ cut out by $|Y_i| \leq 1 - \varepsilon$. Moreover, W_ε is an affinoid domain of $\mathcal{X}^\triangleright$, of affinoid K -algebra $K\{X_1, \dots, X_n, (1 - \varepsilon)^{-1}Y_1, \dots, (1 - \varepsilon)^{-1}Y_m\}/(f_1, \dots, f_r)$.

(2.18) There is a natural *specialization map* $\mathrm{sp}_{\mathcal{X}} : \mathcal{X}^\triangleright \rightarrow \mathcal{X}_0$ which is defined as follows. If $\mathcal{X} = \mathrm{Spf}(A)$ is affine, a point of $\mathcal{X}^\triangleright$ gives rise to a continuous character $\chi_x : A \rightarrow \mathcal{H}(x)^\circ$, where we have denoted by $\mathcal{H}(x)^\circ$ the valuation ring of $\mathcal{H}(x)$, which in turn gives rise to a character $\tilde{\chi}_x : A/I \rightarrow \widehat{\mathcal{H}(x)}$, where I is the largest ideal of definition of A . The kernel of $\tilde{\chi}_x$ is by definition the point $\mathrm{sp}_{\mathcal{X}}(x) \in \mathcal{X}_0 = \mathrm{Spec}(A/I)$. If \mathcal{U} is an open formal subscheme of \mathcal{X} then $\mathcal{U}^\triangleright \cong \mathrm{sp}_{\mathcal{X}}^{-1}(\mathcal{U}_0)$, and the restriction of $\mathrm{sp}_{\mathcal{X}}$ to the latter coincides with $\mathrm{sp}_{\mathcal{U}}$. Therefore, the definition we gave extends to general special formal R -schemes. The map $\mathrm{sp}_{\mathcal{X}}$ is anticontinuous, i.e. the inverse image of an open subset of \mathcal{X} is closed. For example, if $\mathcal{X} = \mathrm{Spf}(A)$ and Z is a closed subset of \mathcal{X} defined by an ideal (f_1, \dots, f_r) , then

$$\mathrm{sp}_{\mathcal{X}}^{-1}(Z) = \{x \in \mathcal{X}^\triangleright \mid |f_i(x)| < 1 \text{ for all } i = 1, \dots, r\}.$$

As in [Ber96b, 0.2.6], $\mathrm{sp}_{\mathcal{X}}$ can be viewed as a morphism of locally ringed sites $\mathrm{sp}_{\mathcal{X}} : \mathcal{X}^\triangleright \rightarrow \mathcal{X}$. Note that this map is often called also *reduction map*.

(2.19) If $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a morphism of special formal k -schemes, then $\mathrm{sp}_{\mathcal{X}} \circ f^\triangleright = f \circ \mathrm{sp}_{\mathcal{Y}}$. Moreover, if Z is a subscheme of \mathcal{X}_0 , then by [Ber96b, 0.2.7] (or [Ber96a, 1.3]) the canonical morphism of formal k -schemes $\widehat{\mathcal{X}/Z} \rightarrow \mathcal{X}$ induces an isomorphism of k -analytic spaces $(\widehat{\mathcal{X}/Z})^\triangleright \cong \mathrm{sp}_{\mathcal{X}}^{-1}(Z)$.

(2.20) Assume from now on that we are working over a trivially valued field k , and let \mathcal{X} be a special formal k -scheme. The closed immersion $\mathcal{X}_0 \rightarrow \mathcal{X}$ gives rise to an immersion $(\mathcal{X}_0)^\triangleright \rightarrow \mathcal{X}^\triangleright$, and we define the *punctured Berkovich space* \mathcal{X}^* of \mathcal{X} as the subspace

$$\mathcal{X}^* = \mathcal{X}^\triangleright \setminus (\mathcal{X}_0)^\triangleright$$

of $\mathcal{X}^\triangleright$. It's a k -analytic space, introduced by Thuillier in [Thu07, 1.7] (where it is called the generic fiber of \mathcal{X}). Any adic morphism of special formal k -schemes $f : \mathcal{Y} \rightarrow \mathcal{X}$ induces a morphism $f^* : \mathcal{Y}^* \rightarrow \mathcal{X}^*$, since we have $(f^\triangleright)^{-1}((\mathcal{X}_0)^\triangleright) \subset (\mathcal{Y}_0)^\triangleright$. We will denote again by $\mathrm{sp}_{\mathcal{X}} : \mathcal{X}^* \rightarrow \mathcal{X}$ the restriction of the specialization map to \mathcal{X}^* .

(2.21) Examples. If \mathcal{X} is of finite type over k , then $\mathcal{X}_0 = \mathcal{X}$, and therefore \mathcal{X}^* is empty. If $\mathcal{X} = \mathrm{Spf}(k[[t]])$, then $\mathcal{X}^* \cong]0, 1[$ is the punctured open unit disc in $\mathbb{A}_k^{1,an}$.

(2.22) If $\mathcal{X} = \mathrm{Spf}(k\{X_1, \dots, X_n\}[[Y_1, \dots, Y_m]]/(f_1, \dots, f_r))$ is an affine special formal k -scheme, we can describe \mathcal{X}^* along the lines of 2.17. The complement in \mathcal{X}^* of the zero locus $V(Y_i)$ of one of the Y_i is the increasing union $\mathcal{X}^* \setminus V(Y_i) = \bigcup_{0 < \varepsilon \leq 1/2} W_{i,\varepsilon}$, where $W_{i,\varepsilon}$ is the subspace of \mathcal{X}^\square cut out by the inequalities $|Y_j| \leq 1 - \varepsilon$ for every j and $\varepsilon \leq |Y_i|$. The subspace $W_{i,\varepsilon}$ is an affinoid domain of \mathcal{X}^* , of affinoid k -algebra

$$\mathcal{W}_{i,\varepsilon} = \frac{k\{X_1, \dots, X_n\}\{\varepsilon Y_i^{-1}, (1 - \varepsilon)^{-1}Y_1, (1 - \varepsilon)^{-1}Y_2, \dots, (1 - \varepsilon)^{-1}Y_m\}}{(f_1, \dots, f_r)}.$$

We then have $\mathcal{X}^* = \bigcup_{i=1}^m (\bigcup_{0 < \varepsilon \leq 1/2} W_{i,\varepsilon})$. Moreover, if we denote by $W_{i,\varepsilon}^\circ$ the open subspace of \mathcal{X}^\square cut out by $|Y_j| < 1 - \varepsilon$ for every j and $\varepsilon > |Y_i|$, then the family $\{W_{i,\varepsilon}^\circ\}_{i=1, \dots, m, 0 < \varepsilon \leq 1/2}$ is an open cover of \mathcal{X}^* . We also denote $W_{i,\varepsilon}$ and $\mathcal{W}_{i,\varepsilon}$ by $W_{Y_i,\varepsilon}$ and $\mathcal{W}_{Y_i,\varepsilon}$ respectively. Note that if t is a nonzero element of the largest ideal of definition of \mathcal{X} , we can analogously write $\mathcal{X}^* \setminus V(t)$ as an increasing union of affinoid domains $\{W_{t,\varepsilon}\}_\varepsilon$.

(2.23) **Fundamental example.** We now discuss in detail what happens in the algebraic case of 2.4, when working over k . Let X be a separated k -scheme of finite type. Then to every point x of X^{an} we can associate a morphism $\varphi_x: \mathrm{Spec}(\mathcal{H}(x)) \rightarrow X$, which sits in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{H}(x)) & \xrightarrow{\varphi_x} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{H}(x)^\circ) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

where $\mathcal{H}(x)^\circ$ is the valuation ring of $\mathcal{H}(x)$. We say that x has center on X if we can fit in the diagram above a morphism $\overline{\varphi}_x: \mathrm{Spec}(\mathcal{H}(x)^\circ) \rightarrow X$ which extends φ_x . By the valuative criterion of separatedness if such an extension exists then it is unique. The center of x on X is then by definition the image in X of the closed point of $\mathrm{Spec}(\mathcal{H}(x)^\circ)$ via $\overline{\varphi}_x$. We denote this point by $\mathrm{sp}_X(x)$, and we write X^\square for the subset of X^{an} consisting of the points which have center on X . The space X^\square can be thought of as a bounded version of X^{an} , and it coincides with the space defined in 2.15 if X is seen as a formal k -scheme of finite type. For example, $(\mathbb{A}_k^n)^\square$ is the closed unit polydisc in $\mathbb{A}_k^{n,an}$. If X is proper, then by the valuative criterion of properness we have $X^\square \cong X^{an}$. Let now Z be a closed subvariety of X and set $\mathcal{X} = \widehat{X/Z}$. Then we have $\mathcal{X}^\square = \mathrm{sp}_X^{-1}(Z)$. Moreover, the restriction of sp_X to \mathcal{X}^\square is the specialization map $\mathrm{sp}_{\mathcal{X}}$ defined in 2.18. The space \mathcal{X}^\square can be thought of as an (infinitesimal) tubular neighborhood of Z^\square in X^\square . Note that Z^{an} is canonically isomorphic to the subspace $\rho^{-1}(Z)$ of X^{an} , where $\rho: X^{an} \rightarrow X$ is the structure morphism defined in 2.12. Similarly, we have $Z^\square = \rho^{-1}(Z) \cap X^\square \subset X^{an}$. Therefore, we have $\mathcal{X}^* = \mathcal{X}^\square \setminus Z^\square = \mathrm{sp}_X^{-1}(Z) \setminus \rho^{-1}(Z)$. In words, \mathcal{X}^* is the set of semivaluations

on X which have center in Z but are not semivaluations on Z . It can be thought of as a *punctured tubular neighborhood* (or *link*) of Z^\triangleright in X^\triangleright .

(2.24) Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be an admissible blowup of special formal k -schemes. Then f induces an isomorphism of punctured spaces $f^* : \mathcal{Y}^* \xrightarrow{\sim} \mathcal{X}^*$. In the algebraic case of 2.23 this follows from the valuative criterion of properness (see [Thu07, 1.11]); the general case is [BBT13, 4.5.1].

We conclude the section by giving definitions of admissibility for special formal k -schemes and for special k -algebras.

(2.25) Let \mathcal{X} be a special formal k -scheme. We say that \mathcal{X} is *admissible* if the canonical morphism of sheaves $\mathcal{O}_{\mathcal{X}} \rightarrow (\mathrm{sp}_{\mathcal{X}})_* \mathcal{O}_{\mathcal{X}^*}$ is a monomorphism.

(2.26) If A is a special k -algebra and J is the largest ideal of definition of A , we define the *torsion ideal* of A as $\mathrm{Tor}_A = \{a \in A \mid a \in \mathrm{Tor}_t(A) \ \forall t \in J\}$, where $\mathrm{Tor}_t(A)$ denotes the t -torsion of A ; Tor_A is then an ideal of A . We say that A is *admissible* if $\mathrm{Tor}_A = 0$.

(2.27) Remarks. If A is an admissible special k -algebra, then the largest ideal of definition of A is nonempty. Therefore, A is not topologically of finite type over k ; the converse holds whenever A is a domain. If $\{g_1, \dots, g_s\}$ is a set of generators of J , then $\mathrm{Tor}_A = \bigcap_{i=1}^s \mathrm{Tor}_{g_i}(A)$, hence A is admissible if and only if the canonical morphism $A \rightarrow \prod_{i=1}^s A[g_i^{-1}]$ is injective. Therefore, if A is an admissible special k -algebra, for every nonzero element f of A the complete localization $A\{f^{-1}\}$ is admissible. Indeed, the same proof as in the finite type case applies, see [Bos14, 2.3.13]. If A is an algebra topologically of finite type over $k[[t]]$, seen as a special k -algebra, then A is admissible if and only if it has no t -torsion. This shows that our definition of admissible algebra coincides with the usual one in this case.

3. NORMALIZED BERKOVICH SPACES OF SPECIAL FORMAL k -SCHEMES

In this section we start by defining an $\mathbb{R}_{>0}$ -action on the punctured Berkovich space \mathcal{X}^* of a special formal k -scheme \mathcal{X} . We then introduce our primary object of study, the Normalized Berkovich space $T_{\mathcal{X}}$ of \mathcal{X} , as the quotient of \mathcal{X}^* by this action.

(3.1) One important characteristic of Berkovich analytic spaces is that they distinguish between equivalent but not equal seminorms. For example, if k is a trivially valued field, γ is an element of $\mathbb{R}_{>0}$, and $|\cdot|_x$ is an element of the closed unit disc $\mathrm{Spf}(k\{T\}) = \mathcal{M}(k\{T\})$ in the analytic affine line $\mathbb{A}_k^{1,an}$, then also $|\cdot|_x^\gamma$ is an element of $\mathcal{M}(k\{T\})$. Indeed, the Banach norm of $k\{T\}$ is the T -adic one with $|T| = 1$, so is the trivial norm; it follows that the elements of $\mathcal{M}(k\{T\})$ are the seminorms $|\cdot|_x$ on $k\{T\}$ satisfying $|f|_x \leq 1$ whenever $f \in k\{T\}$. Then $|\cdot|_x^\gamma$ is multiplicative, trivial on k , it satisfies both the ultrametric inequality and $|f|_x^\gamma \leq 1$ for $f \in k\{T\} \setminus \{0\}$. Observe that the fact that the absolute value of k is trivial, and thus invariant under exponentiation by elements of $\mathbb{R}_{>0}$, is crucial. This leads to the definition of

an action of $\mathbb{R}_{>0}$ on the topological space underlying \mathcal{X}^* . We will first see how this works on another easy example.

(3.2) If $\mathcal{X} = \mathrm{Spf}(k[[T]])$ then \mathcal{X}^\natural is the open unit disc D_- in the analytic affine line $\mathbb{A}_k^{1,an}$ and \mathcal{X}^* is the punctured open unit disc $D_- \setminus \{0\}$. The latter is homeomorphic to the open segment $]0, 1[$, and under this identification $\mathbb{R}_{>0}$ acts on it by exponentiation. More precisely, following 2.22 we can write \mathcal{X}^* as the increasing union of the annuli $A_\varepsilon = \{\varepsilon \leq |T| \leq 1 - \varepsilon\} = \mathcal{M}(\mathcal{A}_\varepsilon) \subset \mathbb{A}_k^{1,an}$ for $0 < \varepsilon \leq 1/2$, where $\mathcal{A}_\varepsilon = k\{\varepsilon T^{-1}, (1 - \varepsilon)^{-1}T\}$. Given γ in $\mathbb{R}_{>0}$ and an element $|\cdot|_x$ of \mathcal{X}^* , a similar computation as in 3.1 shows that, for ε small enough, $|\cdot|_x^\gamma$ is a bounded seminorm on \mathcal{A}_ε . This gives a well defined action of $\mathbb{R}_{>0}$ on \mathcal{X}^* since for $\varepsilon \geq \delta$ the inclusion $A_\varepsilon \hookrightarrow A_\delta$ comes from the identity map $\mathcal{A}_\delta \cong k[[T^{-1}, T]] \rightarrow k[[T^{-1}, T]] \cong \mathcal{A}_\varepsilon$, only the Banach norms change.

The definition of an action of $\mathbb{R}_{>0}$ as left composition with \exp_γ can be extended to the punctured Berkovich space of any affine special formal k -scheme thanks to the following lemma.

(3.3) Lemma. *Let \mathcal{X} be an affine special formal k -scheme. Then there is a unique way to define an action of $\mathbb{R}_{>0}$ on the topological space underlying \mathcal{X}^* so that the following property is satisfied: if V is an affinoid domain of \mathcal{X}^* of affinoid algebra \mathcal{V} , $x: \mathcal{V} \rightarrow \mathbb{R}_{\geq 0}$ is a point of V and γ is an element of $\mathbb{R}_{>0}$ such that the seminorm x^γ is a point of V , then $\gamma \cdot x = x^\gamma$.*

Proof. We begin by showing that for every $x \in \mathcal{X}^*$ and $\gamma \in \mathbb{R}_{>0}$ there exists an affinoid domain $V \subset \mathcal{X}^*$ such that both x and x^γ belong to V . To do so, we pick an element t in the largest ideal of definition of \mathcal{V} such that t does not vanish on x . In the notation of 2.22, there exists $\varepsilon > 0$ such that $x \in W_{t,\varepsilon}$, and by further shrinking ε as before we can find a bigger affinoid domain $W_{t,\delta}$ containing both x and x^γ , i.e. such that x^γ is bounded on $W_{t,\delta}$. If $V = \mathcal{M}(\mathcal{V})$ is an affinoid domain of \mathcal{X}^* such that x and x^γ are bounded seminorms on \mathcal{V} , we write $\gamma \cdot_V x$ for the point of \mathcal{X}^* corresponding to the seminorm x^γ . We then have to show that the point $\gamma \cdot_V x$ does not depend on the choice of the affinoid V . Let us first assume that V and W are two such affinoid domains, of affinoid algebras \mathcal{V} and \mathcal{W} respectively, and that we have an inclusion $\iota: W \hookrightarrow V$. Denote by $\varphi: \mathcal{V} \rightarrow \mathcal{W}$ the corresponding morphism of affinoid algebras. Then the fact that x belongs to both V and W amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{x} & \mathbb{R}_{\geq 0} \\ \varphi \downarrow & & \parallel \\ \mathcal{W} & \xrightarrow{x} & \mathbb{R}_{\geq 0} \end{array}$$

By composing with \exp_γ we get the following diagram,

$$\begin{array}{ccccc}
 & & \gamma \cdot_V x & & \\
 & \nearrow & & \searrow & \\
 \mathcal{V} & \xrightarrow{x} & \mathbb{R}_{\geq 0} & \xrightarrow{\exp_\gamma} & \mathbb{R}_{\geq 0} \\
 \downarrow \varphi & & & & \parallel \\
 \mathcal{W} & \xrightarrow{x} & \mathbb{R}_{\geq 0} & \xrightarrow{\exp_\gamma} & \mathbb{R}_{\geq 0} \\
 & \searrow & & \nearrow & \\
 & & \gamma \cdot_W x & &
 \end{array}$$

whose commutativity implies that $\iota(\gamma \cdot_W x) = \gamma \cdot_V x$, i.e. $\gamma \cdot_W x$ and $\gamma \cdot_V x$ are the same point of \mathcal{X}^* . It is now enough to show that if δ and V are as above, i.e. if both $\gamma \cdot_{W_{t,\delta}} x$ and $\gamma \cdot_V x$ can be defined, then there is an affinoid domain W of \mathcal{X}^* which is contained both in $W_{t,\delta}$ and in V and such that $\gamma \cdot_W x$ can be defined. Since $\mathcal{X}^\triangleright$ is covered by the open subspaces defined by the inequalities $|Y_i| < 1 - \varepsilon$, V being compact is contained in one of those and so it is an affinoid subdomain of the affinoid of $\mathcal{X}^\triangleright$ defined by the inequalities $|Y_i| \leq 1 - \varepsilon$. By further shrinking δ , we can assume that $W = V \cap W_{t,\delta}$ is the subspace of V defined by $|t| > \delta$. Therefore W is an affinoid subdomain of V , of affinoid algebra $\mathcal{W} = \mathcal{V}\{\delta t^{-1}\}$. It remains to show that if we see x as a bounded seminorm on \mathcal{W} , then x^γ is again bounded on \mathcal{W} . A general element of \mathcal{W} can be written as a/t^n for some a in \mathcal{V} and some $n \in \mathbb{N}$, and its norm $\|a/t^n\|_{\mathcal{W}}$ is equal to $\|a\|_{\mathcal{V}} \varepsilon^{-n}$, where $\|a\|_{\mathcal{V}}$ is the norm of a in \mathcal{V} . We then have $x^\gamma(a/t^n) = \frac{x(a)^\gamma}{x(t)^{n\gamma}} \leq \frac{C\|a\|_{\mathcal{V}}}{\varepsilon^n}$, where we used both the fact that x^γ is bounded on \mathcal{V} (with $C > 0$ a constant such that $x(b) \leq C\|b\|_{\mathcal{V}}$ for every b in \mathcal{V}) and the fact that $x^\gamma(t) \geq \varepsilon$ on $\mathcal{W}_{t,\delta}$. This proves that x^γ is bounded on \mathcal{V} (with same constant C realizing the bound), concluding the proof. \square

This result allows us to extend the definition of the $\mathbb{R}_{>0}$ -action to the Berkovich space of any special formal scheme. If $\mathcal{X} = \bigcup_i \mathrm{Spf}(A_i)$, by covering $(\mathrm{Spf} A_i)^* \cap (\mathrm{Spf} A_j)^*$ with affinoid domains we see that the $\mathbb{R}_{>0}$ -actions on $(\mathrm{Spf} A_i)^*$ and on $(\mathrm{Spf} A_j)^*$ coincide on the intersection, and that the action we defined does not depend on a chosen presentation. We summarize the results obtained above in the following proposition.

(3.4) Proposition. *Let \mathcal{X} be a special formal scheme over k . Then there is a unique way to define an action of $\mathbb{R}_{>0}$ on the topological space underlying \mathcal{X}^* which restricts on \mathcal{U}^* to the $\mathbb{R}_{>0}$ -action described in Lemma 3.3 whenever \mathcal{U} is an affine open formal subscheme of \mathcal{X} .*

(3.5) Remark. The $\mathbb{R}_{>0}$ -action on \mathcal{X}^* is free (i.e. the orbits $\mathbb{R}_{>0} \cdot x$ are in bijection with $\mathbb{R}_{>0}$). Indeed, either $\mathbb{R}_{>0} \cdot x \cong \mathbb{R}_{>0}$ or $\mathbb{R}_{>0} \cdot x = \{x\}$; the second is equivalent to x being a trivial absolute value, but all trivial absolute values of $\mathcal{X}^\triangleright$ lie in \mathcal{X}_0^{an} .

(3.6) Corollary. *Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be an adic morphism of special formal k -schemes. Then the induced morphism $f^* : \mathcal{Y}^* \rightarrow \mathcal{X}^*$ is equivariant.*

Proof. Covering \mathcal{Y}^* and \mathcal{X}^* by suitable affinoid domains we can see f^* locally as a morphism of affinoid domains; Lemma 3.3 allows us to conclude. \square

(3.7) Putting together all the previous results, we deduce that the association $\mathcal{X} \mapsto \mathcal{X}^*$ gives a functor from the category of special formal k -schemes with adic morphisms to the category of k -analytic spaces with a free $\mathbb{R}_{>0}$ -action on the underlying topological space and equivariant analytic morphisms.

We now consider the quotient of \mathcal{X}^* by the $\mathbb{R}_{>0}$ -action.

(3.8) Let \mathcal{X} be a special formal scheme over k . Denote by $T_{\mathcal{X}}$ the quotient of the space \mathcal{X}^* by the action of $\mathbb{R}_{>0}$, and by $\pi : \mathcal{X}^* \rightarrow T_{\mathcal{X}}$ the quotient map. We endow $T_{\mathcal{X}}$ with both the quotient topology and the quotient G -topology. The latter is defined as follows: we declare that a subspace U of $T_{\mathcal{X}}$ is G -open if $\pi^{-1}(U)$ is G -open for the G -topology of \mathcal{X}^* , and that a family $\{U_i\}_i$ of subspaces of U is a G -covering of U if $\{\pi^{-1}(U_i)\}_i$ is a G -covering of $\pi^{-1}(U)$. It is easy to verify that this defines a G -topology on $T_{\mathcal{X}}$, which is finer than the quotient topology because the G -topology on \mathcal{X}^* is finer than the Berkovich topology. We call *analytic domains* of $T_{\mathcal{X}}$ its G -open subspaces. To provide $T_{\mathcal{X}}$ with the structure of a ringed G -topological space in k -algebras, we endow it with the sheaf $\mathcal{O}_{T_{\mathcal{X}}} = \pi_* \mathcal{O}_{\mathcal{X}^*}$, the push-forward of the sheaf of analytic functions on \mathcal{X}^* via the projection map.

(3.9) If $f \in \mathcal{O}_T(V)$ is a function on $V \subset T_{\mathcal{X}}$, it doesn't make sense to evaluate f in a point of V . Nevertheless, it follows from Lemma 3.3 that it makes sense to ask whether this value lies in $\{0\}$, $\{1\}$, $]0, 1[$ or $]1, \infty[$, since these sets are the orbits of the action of $\mathbb{R}_{>0}$ on $\mathbb{R}_{>0}$ by exponentiation. In particular, the sheaf $\pi_* \mathcal{O}_{\mathcal{X}^*}^{\circ}$, which is a subsheaf of $\mathcal{O}_{T_{\mathcal{X}}}$, can really be thought of as the *sheaf of bounded by 1 analytic functions* on $T_{\mathcal{X}}$, and the sheaf $\pi_* \mathcal{O}_{\mathcal{X}^*}^{\circ\circ}$ can be thought of as the *sheaf of strictly bounded by 1 analytic functions* on $T_{\mathcal{X}}$. We denote these sheaves by $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}$ and $\mathcal{O}_{T_{\mathcal{X}}}^{\circ\circ}$ respectively.

(3.10) **Remark.** It also makes sense to evaluate in points of $T_{\mathcal{X}}$ every function which is constant on the orbits of points for the $\mathbb{R}_{>0}$ -action. For example, if f and g are functions on $V \subset T_{\mathcal{X}}$ and x is a point of V where f and g do not vanish, then we can evaluate $\log |f| / \log |g|$ at x . Note that this value is encoded in the structure of the normalized space. For example, if both f and g take values in $]0, 1[$, $(\log |f| / \log |g|)(x)$ can be defined as $\sup\{a/b \mid a, b \in \mathbb{N}, b \neq 0 \text{ and } |f^b(x)| \leq |g^a(x)|\}$; the other cases are similar.

(3.11) **Lemma.** *The projection $\pi : \mathcal{X}^* \rightarrow T_{\mathcal{X}}$ is an open map, and the topological space underlying $T_{\mathcal{X}}$ is Hausdorff.*

Proof. The projection $\pi : \mathcal{X}^* \rightarrow T_{\mathcal{X}}$ is an open map because I acts on \mathcal{X}^* by homeomorphisms. This follows from the definition of the Berkovich topology on \mathcal{X}^* and the continuity of $x \mapsto x^\lambda$. Now, if x and y are two points not in the same I -orbit, since \mathcal{X}^* is Hausdorff we can pick an affinoid of \mathcal{X}^* containing both. Since x and y are not in the same orbit, we can find two functions f and g on this affinoid, not vanishing on x and y , and

such that $\log |f(x)|/\log |g(x)| \neq \log |f(y)|/\log |g(y)|$. Because the quotients of the logarithms are continuous, the orbit equivalence relation is closed in $\mathcal{X}^* \times \mathcal{X}^*$. Therefore, \mathcal{X}^* being Hausdorff, $T_{\mathcal{X}}$ is Hausdorff as well. \square

(3.12) Lemma. *Both the G -sheaf $\mathcal{O}_{T_{\mathcal{X}}}$ and its restriction to the usual topology of $T_{\mathcal{X}}$ are local sheaves.*

Proof. Let x be a point of $T_{\mathcal{X}}$. Then $\mathcal{I} = \{f \in \mathcal{O}_{T_{\mathcal{X}},x} \text{ s.t. } |f(x)| = 0\}$ is an ideal of $\mathcal{O}_{T_{\mathcal{X}},x}$, where we write $\mathcal{O}_{T_{\mathcal{X}},x}$ for the local ring for the usual topology. If $|f(x)| \neq 0$ then f , seen as a function on a neighborhood of $\pi^{-1}(x)$ in \mathcal{X}^* , doesn't vanish in any point of some open neighborhood U of some point of $\pi^{-1}(x)$. Therefore f has no zero, and is hence invertible, on the $\mathbb{R}_{>0}$ -invariant subspace $\pi^{-1}(\pi(U))$, which is open by 3.11. This proves that \mathcal{I} is the unique maximal ideal of $\mathcal{O}_{T_{\mathcal{X}},x}$, and so the restriction of $\mathcal{O}_{T_{\mathcal{X}},x}$ to the usual topology is local. Since the G -topology of $T_{\mathcal{X}}$ is finer than its usual topology, the rest of the statement follows. \square

(3.13) Example. If \mathcal{X} is $\text{Spf}(\mathbb{C}[[X, Y]])$, the completion of the complex affine plane $\mathbb{A}_{\mathbb{C}}^2$ at the origin, the topological space $T_{\mathcal{X}}$ is canonically homeomorphic to the *valuative tree* T introduced by Favre and Jonsson in [FJ04]. The valuative tree is defined as the set of (semi-)valuations v on $\mathbb{C}[[X, Y]]$ extending the trivial valuation on \mathbb{C} and such that $\min\{v(X), v(Y)\} = 1$, endowed with the topology it inherits from the Berkovich space \mathcal{X}^* via the inclusion which sends a valuation $v \in T$ to the seminorm e^{-v} . The restriction of the projection $\pi : \mathcal{X}^* \rightarrow T_{\mathcal{X}}$ to T is then a continuous bijection, and it is open since π is open by 3.11, therefore it is a homeomorphism.

(3.14) Example. More generally, in the algebraic case discussed in Examples 2.4 and 2.23, i.e. when $\mathcal{X} = \widehat{X/Z}$ is the formal completion of a k -variety X along a closed subvariety Z , the normalized Berkovich space $T_{\mathcal{X}}$ can be interpreted as the *normalized non-archimedean link* of Z in X . The topological space underlying $T_{\mathcal{X}}$ can be described explicitly as the space of normalized valuations on X which are centered on Z but are not valuations on Z , i.e. $T_{\mathcal{X}} = (\text{sp}_X^{-1}(Z) \setminus Z^{\triangleright}) / \mathbb{R}_{>0}$. An explicit normalization can be given as follows. Let \mathcal{I} be the coherent ideal sheaf of X defining Z , and for each element x of \mathcal{X}^* set $x(\mathcal{I}) = \max\{x(f) \mid f \in \mathcal{I}_{\text{sp}_{\mathcal{X}}(x)}\} > 0$, where $\mathcal{I}_{\text{sp}_{\mathcal{X}}(x)}$ denotes the stalk of \mathcal{I} at $\text{sp}_{\mathcal{X}}(x)$. Then, as in 3.13, since for every γ in $\mathbb{R}_{>0}$ and x in \mathcal{X}^* we have $\gamma \cdot x(\mathcal{I}) = x(\mathcal{I})^{\gamma}$ the restriction of π to the subspace $\{x \in X^{\triangleright} \mid x(\mathcal{I}) = 1/e\}$ of \mathcal{X}^* is a homeomorphism onto $T_{\mathcal{X}}$. Therefore in this case the topological space we consider is similar to the *normalized valuation space* considered in [BdFFU13, §2.2]; the difference is that they consider only valuations with trivial kernel.

(3.15) Let \mathcal{X} be a special formal k -scheme. Then the specialization map on \mathcal{X}^* induces an anticontinuous map $\text{sp}_{\mathcal{X}} : T_{\mathcal{X}} \rightarrow \mathcal{X}$, which we call again *specialization*. Indeed, the valuation ring of two elements of \mathcal{X}^* in the same $\mathbb{R}_{>0}$ -orbit is the same, therefore the specialization map on \mathcal{X}^* passes to the quotient, inducing a map of sets $\text{sp}_{\mathcal{X}} : T_{\mathcal{X}} \rightarrow \mathcal{X}$ such

$\mathrm{sp}_{\mathcal{X}} \circ \pi = \mathrm{sp}_{\mathcal{X}}$. Anticontinuity follows from the fact that the specialization on \mathcal{X}^* is anticontinuous and π is open by Lemma 3.11. We will prove in 4.21 that if \mathcal{X} is admissible then the specialization map is surjective.

(3.16) To study the spaces we have been considering so far, it is convenient to temporarily introduce a suitable category. We denote by \mathcal{C} the category whose objects are the triples $(T, \mathcal{O}_T, \mathcal{O}_T^\circ)$, where T is a topological space endowed with an additional finer G -topology, \mathcal{O}_T is a local sheaf in k -algebras on T for the G -topology, and \mathcal{O}_T° is a subsheaf in k -algebras of \mathcal{O}_T ; and such that a morphism $(T, \mathcal{O}_T, \mathcal{O}_T^\circ) \rightarrow (T', \mathcal{O}_{T'}, \mathcal{O}_{T'}^\circ)$ is given by a continuous and G -continuous map $f : T \rightarrow T'$ and a local morphism of sheaves $f_\# : \mathcal{O}_{T'} \rightarrow f_* \mathcal{O}_T$ such that $f(\mathcal{O}_{T'}^\circ) \subset f_* \mathcal{O}_T^\circ$. We will always write simply \mathcal{O}_T (respectively \mathcal{O}_T°) also for the restriction of \mathcal{O}_T (resp. \mathcal{O}_T°) to the topology of T and, when no risk of confusion will arise, we will write T for an object $(T, \mathcal{O}_T, \mathcal{O}_T^\circ)$ of \mathcal{C} .

(3.17) Let \mathcal{X} be a special formal scheme over k . We define the *normalized Berkovich space* of \mathcal{X} as the object $T_{\mathcal{X}} = (T_{\mathcal{X}}, \mathcal{O}_{T_{\mathcal{X}}}, \mathcal{O}_{T_{\mathcal{X}}}^\circ)$ of \mathcal{C} . This gives a functor $T : (SFor_k) \rightarrow \mathcal{C}$ from the category of special formal k -schemes with adic morphisms to \mathcal{C} . In Section 6 we will investigate the properties of the functor T and determine its essential image.

(3.18) Remark. Thuillier proved in [Thu07] that whenever k is perfect, X is a k -variety with singular locus Z and $\mathcal{X} = \widehat{X/Z}$, the homotopy type of \mathcal{X}^* is the same as the homotopy type of the dual complex $\mathrm{Dual}(D)$ of the exceptional divisor D of a log resolution Y of X . Using toroidal methods, he constructs an embedding of $\mathrm{Dual}(D) \times \mathbb{R}_{>0}$ into \mathcal{Y}^* and a deformation retract of the latter onto the former, where $\mathcal{Y} = \widehat{Y/D}$. The map $\mathrm{Dual}(D) \times \mathbb{R}_{>0} \hookrightarrow \mathcal{Y}^*$ defined by Thuillier induces an embedding $\mathrm{Dual}(D) \hookrightarrow T_{\mathcal{Y}}$, and analogously we obtain a deformation retraction of $T_{\mathcal{Y}}$ onto a copy of $\mathrm{Dual}(D)$. Since $T_{\mathcal{Y}} \cong T_{X,Z}$, we deduce that the homotopy type of $T_{X,Z}$ is the same as the homotopy type of $\mathrm{Dual}(D)$. Note that by [Kol13] the homotopy type of $\mathrm{Dual}(D)$ can be almost arbitrary. However, by [dFKX12] $\mathrm{Dual}(D)$ is contractible for a wide class of singularities, namely isolated log terminal singularities (in particular, for all toric or finite quotient singularities).

(3.19) We also have a forgetful functor for $: (An_{k((t))}) \rightarrow \mathcal{C}$ sending a $k((t))$ -analytic space X to the triple $(X, \mathcal{O}_X, \mathcal{O}_X^\circ)$.

4. LOCAL ANALYTIC STRUCTURE

In this section we prove one of the main properties of normalized spaces of special formal k -schemes. Although those spaces are not analytic spaces themselves, as locally ringed spaces in k -algebras they are locally isomorphic to analytic spaces defined over some Laurent series field $k((t))$. This is the content of Corollary 4.10. This result is conceptually similar to those discussed in [BBT13, §4.2, 4.3, 4.4, 4.6]. We deduce an analogue for normalized spaces of a theorem of de Jong (Corollary 4.14), and a characterization of

admissible special formal k -schemes (Proposition 4.19). We also prove that the specialization map is surjective for the normalized space of an admissible special formal k -scheme (Theorem 4.21). We first need some results about the Berkovich spaces associated to affine special formal k -schemes.

(4.1) Lemma. *Let \mathcal{X} be an affine special formal scheme over k , let t be a nonzero element of the largest ideal of definition of \mathcal{X} and let V be an affinoid domain of \mathcal{X}^* such that t has no zero on V . Denote by $\mathbb{R}_{>0} \cdot V$ the set of translates of V in \mathcal{X}^* under the $\mathbb{R}_{>0}$ -action. Then we can write $\mathbb{R}_{>0} \cdot V$ as a finite union $\cup_i V_i$, with each V_i stable under the action of $\mathbb{R}_{>0}$ and such that $V_{i,\varepsilon} := V_i \cap W_{t,\varepsilon}$ is a strict affinoid subdomain of $W_{t,\varepsilon}$ for ε small enough. In particular, each V_i is an increasing union of affinoid domains of \mathcal{X}^* , and $\mathbb{R}_{>0} \cdot V$ is an analytic domain of \mathcal{X}^* .*

Proof. As in 2.22, the increasing family $\{W_{t,\varepsilon}^\circ\}_\varepsilon$ is an open cover of $\mathcal{X}^* \setminus V(t)$, so since V is compact V is contained in $W_{t,\delta}^\circ$ for some δ . It follows that V is an affinoid subdomain of $W_{t,\delta}$. Then, by Gerritzen-Grauert theorem (proofs valid in the case of a trivially valued field are given in [Duc03] and [Tem05]), V is a finite union $V = \cup V'_i$ of rational domains of $W_{t,\delta}$. Each V'_i is by definition determined in $W_{t,\delta}$ by finitely many inequalities $|f_j| \leq r_j |g_j|$, for some $f_j, g_j \in \mathcal{W}_{t,\delta}$, $r_j \geq 0$. Then for every $\varepsilon \leq \delta$ we have

$$\begin{aligned} (\mathbb{R}_{>0} \cdot V'_i) \cap W_{t,\varepsilon} &= \\ &= \{x \in W_{t,\varepsilon} \text{ s.t. } |f_j(x)| \leq r_j^\lambda |g_j(x)| \text{ for some } \lambda \in \mathbb{R}_{>0} \text{ and all } j\} \\ &= \{x \in W_{t,\varepsilon} \text{ s.t. } |f_j(x)| \leq |g_j(x)| \text{ for all } j \text{ such that } r_j = 1\}, \end{aligned}$$

which is a strict affinoid subdomain of $W_{t,\varepsilon}$. Therefore, if we set $V_i := \mathbb{R}_{>0} \cdot V'_i$ and $V_{i,\varepsilon} = V_i \cap W_{t,\varepsilon}$ then the V_i and $V_{i,\varepsilon}$ satisfy our requirements. Finally, $\mathbb{R}_{>0} \cdot V$ is an analytic domain of \mathcal{X}^* , G -covered by the affinoids $V_i \cap W_{t,\varepsilon}$. Indeed, every point of $\mathbb{R}_{>0} \cdot V$ is contained in the interior of one set of the form $\mathbb{R}_{>0} \cdot V \cap W_{t,\varepsilon}$, and the latter is the finite union of the affinoids $V_i \cap W_{t,\varepsilon}$. \square

(4.2) Corollary. *Let \mathcal{X} be an affine special formal scheme over k , let t be a nonzero element of the largest ideal of definition of \mathcal{X} and let U be a subset of $\mathcal{X}^* \setminus V(t)$ stable under the action of $\mathbb{R}_{>0}$. Then U is an analytic domain of \mathcal{X}^* if and only if we can write it as a union $U = \cup_i U_i$ in such a way that each U_i is stable under the action of $\mathbb{R}_{>0}$ and is an increasing union $U_i = \cup U_{i,\varepsilon}$ for ε small enough, with $U_{i,\varepsilon}$ a strict affinoid subdomain of $W_{t,\varepsilon}$, and $\{U_{i,\varepsilon}\}_{i,\varepsilon}$ is a G -covering of U .*

Proof. If U is an analytic domain of \mathcal{X}^* then it is G -covered by the affinoid domains that it contains, and applying Lemma 4.1 to each of them we get the decomposition that we want. The converse implication is obvious, since U is by definition G -covered by the affinoids $U_{i,\varepsilon}$. \square

(4.3) For $r \in]0, 1[$, we denote by K_r the affinoid k -algebra $k\{rt^{-1}, r^{-1}t\}$; it is the completed residue field $\mathcal{H}(r)$ of the point r of $\mathrm{Spf}(k[[t]])^* \cong]0, 1[$, and an easy computation shows that it is the field $k((t))$ with the t -adic

absolute value such that $|t| = r$. For $0 < \varepsilon < 1/2$, the k -algebras \mathcal{A}_ε defined in 3.2 are also isomorphic to the field $k((t))$, but their Banach norms are not t -adic (they are not even absolute values). If $r \in [\varepsilon, 1 - \varepsilon]$, the identity map $\mathcal{A}_\varepsilon \rightarrow K_r$ is a bounded morphism of Banach k -algebras. Its boundedness is easy to check algebraically; geometrically this corresponds to the inclusion of the point r , into A_ε , seen as the annulus $[\varepsilon, 1 - \varepsilon]$ in $\mathrm{Spf}(k[[t]])^* \cong]0, 1[$. Nevertheless, note that despite having different Banach norms K_r and \mathcal{A}_ε are isomorphic not only as k -algebras, but also as topological k -algebras, since the neighborhoods of zero in both algebras coincide. This has as a very important consequence the following result.

(4.4) Lemma. *Let \mathcal{B} be a strict affinoid algebra over \mathcal{A}_ε and let r be an element of $[\varepsilon, 1 - \varepsilon]$. Then the canonical morphism $\mathcal{B} \rightarrow \mathcal{B} \hat{\otimes}_{\mathcal{A}_\varepsilon} K_r$ is an isomorphism of k -algebras.*

Proof. Since \mathcal{A}_ε is a field, the strict affinoid \mathcal{A}_ε -algebra \mathcal{B} is of the form $\mathcal{B} = \mathcal{A}_\varepsilon\{X_1, \dots, X_n\}/I$. If $\mathcal{B} = \mathcal{A}_\varepsilon\{X_1, \dots, X_n\}$, an elementary computation shows that the convergence conditions for a series of the form $\sum_{i_0, I} a_{i_0, I} t^{i_0} \underline{X}^I$, for $a_{i_0, I} \in k$, to belong to either \mathcal{B} or $\mathcal{B} \hat{\otimes}_{\mathcal{A}_\varepsilon} K_r$ are the same. Therefore $\mathcal{B} \otimes_{\mathcal{A}_\varepsilon} K_r$, which is isomorphic to \mathcal{B} as a k -algebra, is already complete with respect to the tensor product seminorm, and hence coincides with $\mathcal{B} \hat{\otimes}_{\mathcal{A}_\varepsilon} K_r$. In the general case, $\mathcal{B} = \mathcal{A}_\varepsilon\{X_1, \dots, X_n\}/I$, to show that $\mathcal{B} \otimes_{\mathcal{A}_\varepsilon} K_r$ is complete observe that $\frac{\mathcal{A}_\varepsilon\{X_1, \dots, X_n\}}{I} \otimes_{\mathcal{A}_\varepsilon} K_r \cong \frac{\mathcal{A}_\varepsilon\{X_1, \dots, X_n\} \otimes_{\mathcal{A}_\varepsilon} K_r}{I}$ as normed algebras. Indeed, they are canonically isomorphic as k -algebras, and the fact that the norms are the same can be easily checked explicitly. The algebra on the right hand side is complete since it is an admissible quotient of a Banach algebra by a closed ideal, hence $\mathcal{B} \otimes_{\mathcal{A}_\varepsilon} K_r$ is complete. \square

(4.5) Remark. The result above can fail if the affinoid \mathcal{A}_ε -algebra \mathcal{B} is not strict. For example, if $0 < r < s < \varepsilon \leq 1/2$, then $K_r \hat{\otimes}_{\mathcal{A}_\varepsilon} K_s = 0$. Indeed, the tensor product seminorm on $K_r \otimes_{\mathcal{A}_\varepsilon} K_s$ is the zero seminorm since the element $1 \otimes 1$ of the tensor product is equivalent to $t^n \otimes t^{-n}$ for every $n \in \mathbb{N}$, so $|1 \otimes 1| \leq r^n s^{-n}$, and the latter goes to zero as n goes to infinity.

(4.6) If \mathcal{X}_t is a special formal scheme over $k[[t]]$ then it can be seen as a special formal scheme \mathcal{X} over k , so we get a morphism of formal k -schemes $\mathcal{X} \rightarrow \mathrm{Spf}(k[[t]])$ and therefore a morphism of k -analytic spaces $f : \mathcal{X}^\natural \rightarrow \mathrm{Spf}(k[[t]])^\natural$. If r is any point of $\mathrm{Spf}(k[[t]])^\natural \cong]0, 1[$, then we can consider the fiber product of \mathcal{X}^\natural with the point r in the category of analytic spaces over $\mathrm{Spf}(k[[t]])^\natural$ (see [Ber93]). This analytic space is defined over the non-archimedean field $\mathcal{H}(r)$, which coincides with k whenever $r = 0$ and is otherwise isomorphic to the field $k((t))$ with the t -adic absolute value such that $|t| = r$. The topological space underlying this fiber we consider is canonically homeomorphic to the topological fiber: $f^{-1}(r) \cong \mathcal{X}^\natural \times_{\mathrm{Spf}(k[[t]])^\natural} \mathcal{M}(\mathcal{H}(r))$.

(4.7) Lemma. *Let \mathcal{X}_t and $f : \mathcal{X}^\natural \rightarrow \mathrm{Spf}(k[[t]])^\natural$ be as above, choose $0 < r < 1$ and endow $k((t))$ with the t -adic absolute value such that $|t| = r$. Then $f^{-1}(r)$ is isomorphic to \mathcal{X}_t^\natural as a $k((t))$ -analytic space.*

Proof. We follow the lines of [Nic11, 4.3]. Since $\mathcal{X}^\triangleright$ is G -covered by the Berkovich spaces of the affine formal subschemes of \mathcal{X}_t , we can assume that \mathcal{X}_t is affine, of $k[[t]]$ -algebra $k[[t]]\{X_1, \dots, X_n\}[[Y_1, \dots, Y_m]]/I$. Then following 2.17 we can write $\mathcal{X}_t^\triangleright$ as an increasing union $\mathcal{X}_t^\triangleright = \bigcup_{0 < \varepsilon \leq 1/2} U_\varepsilon$, where U_ε is the subspace of $\mathcal{X}_t^\triangleright$ cut out by the inequalities $|Y_i| \leq 1 - \varepsilon$; it is an affinoid domain of affinoid $k((t))$ -algebra

$$\mathcal{U}_\varepsilon = \frac{k((t))\{X_1, \dots, X_n\}\{(1 - \varepsilon)^{-1}Y_1, \dots, (1 - \varepsilon)^{-1}Y_m\}}{I}.$$

Similarly, by 2.22 we can write $\mathcal{X}^* \setminus V(t)$ as an increasing union $\mathcal{X}^* \setminus V(t) = \bigcup_{0 < \varepsilon \leq 1/2} W_{t,\varepsilon}$, where $W_{t,\varepsilon}$ is the subspace of \mathcal{X}^* cut out by $\varepsilon \leq |t| \leq 1 - \varepsilon$ and $|Y_i| \leq 1 - \varepsilon$; it is an affinoid domain of \mathcal{X}^* of affinoid k -algebra

$$\mathcal{W}_{t,\varepsilon} = \frac{k\{\varepsilon t^{-1}, (1 - \varepsilon)^{-1}t\}\{X_1, \dots, X_n\}\{(1 - \varepsilon)^{-1}Y_1, \dots, (1 - \varepsilon)^{-1}Y_m\}}{I}.$$

Then, for every ε such that $\varepsilon < r < 1 - \varepsilon$, we get a map $U_\varepsilon \rightarrow W_{t,\varepsilon}$ which is a homeomorphism onto $f^{-1}(r) \cap W_{t,\varepsilon}$. These maps glue to a homeomorphism $\mathcal{X}_t^\triangleright \rightarrow f^{-1}(r)$. This homeomorphism is an isomorphism of $k((t))$ -analytic spaces because, since $f^{-1}(r) \cap W_{t,\varepsilon}$ can be identified with the subspace of $f^{-1}(r)$ defined by the inequalities $|Y_i| \leq 1 - \varepsilon$, which is an affinoid domain of affinoid $k((t))$ -algebra $\mathcal{W}_{t,\varepsilon} \hat{\otimes}_{\mathcal{A}_\varepsilon} \mathcal{H}(r) \cong \mathcal{U}_\varepsilon$, it identifies the affinoid domains of $\mathcal{X}_t^\triangleright$ to the ones of $f^{-1}(r)$. \square

(4.8) The morphism $\mathcal{X} \rightarrow \mathrm{Spf}(k[[t]])$ of point 4.6 is adic if and only if \mathcal{X}_t is locally of finite type over $k[[t]]$, so in general it does not induce a morphism between punctured Berkovich spaces. However, we get an $\mathbb{R}_{>0}$ -equivariant morphism of k -analytic spaces $f|_{\mathcal{X}^\triangleright \setminus V(t)} : \mathcal{X}^\triangleright \setminus V(t) \rightarrow \mathrm{Spf}(k[[t]])^*$. Since $f|_{\mathcal{X}^\triangleright \setminus V(t)}$ is equivariant and $\mathrm{Spf}(k[[t]])^* \cong]0, 1[$ consists of a unique $\mathbb{R}_{>0}$ -orbit, for any $r \in]0, 1[$ we have that as topological spaces $f^{-1}(r) \cong (\mathcal{X}^\triangleright \setminus V(t)) / (\mathbb{R}_{>0})$, and the latter is homeomorphic to $T_{\mathcal{X}} \setminus V(t)$. As a consequence of the results above, we have the following important result.

(4.9) Theorem. *Let \mathcal{X}_t be a special formal scheme over $k[[t]]$. Then $\pi|_{f^{-1}(r)} : f^{-1}(r) \rightarrow T_{\mathcal{X}} \setminus V(t)$ induces an isomorphism between $\mathrm{for}(\mathcal{X}_t^\triangleright)$ and $T_{\mathcal{X}} \setminus V(t)$ in \mathcal{C} .*

Proof. Without loss of generality we can assume that \mathcal{X}_t is affine. We will prove that the continuous map $\varphi : \mathcal{X}_t^\triangleright \xrightarrow{\sim} T_{\mathcal{X}} \setminus V(t)$ obtained from Lemma 4.7 and paragraph 4.8 can be upgraded to an isomorphism in \mathcal{C} . Observe that, once $\mathcal{X}_t^\triangleright$ is identified with $f^{-1}(r)$, φ is induced by the projection π , and therefore it is automatically a morphism of locally ringed G -topological spaces. For $0 < \varepsilon < 1/2$, the map $f : \mathcal{X}^\triangleright \rightarrow \mathrm{Spf}(k[[t]])^\triangleright$ of 4.6 induces a surjective morphism from $W_{t,\varepsilon}$ to the affinoid domain $A_\varepsilon = \mathcal{M}(\mathcal{A}_\varepsilon) \cong [\varepsilon, 1 - \varepsilon]$ of $(\mathrm{Spf}(k[[t]]))^\triangleright \cong]0, 1[$; the corresponding morphism $\mathcal{A}_\varepsilon \rightarrow \mathcal{W}_{t,\varepsilon}$ is the unique k -morphism sending t to t , and it endows $\mathcal{W}_{t,\varepsilon}$ with the structure of a strict affinoid algebra over \mathcal{A}_ε . To see that φ is G -continuous we have to show

that $\pi^{-1}(U) \cap \mathcal{X}_t^\triangleright$ is an analytic domain of $\mathcal{X}_t^\triangleright$ whenever $U \subset T\mathcal{X} \setminus V(t)$ is such that $\pi^{-1}(U)$ is an analytic domain of \mathcal{X}^* . Using 4.2, we write $\pi^{-1}(U)$ as $\cup U_i$, with $U_{i,\varepsilon} = U_i \cap W_{t,\varepsilon}$ strict affinoid subdomain of $W_{t,\varepsilon}$ for every i and every ε small enough. Following the isomorphism of 4.7, we deduce that $\pi^{-1}(U) \cap \mathcal{X}_t^\triangleright$ is G -covered by the affinoid domains $U_{i,\varepsilon} \times_{A_\varepsilon} \mathcal{M}(K_r)$ of $\mathcal{X}_t^\triangleright$, and is therefore an analytic domain of $\mathcal{X}_t^\triangleright$. Moreover, if we choose ε small enough, each $U_{i,\varepsilon}$, being strict over $W_{t,\varepsilon}$ which is strict over A_ε , is itself strict over A_ε . We deduce that, as k -algebras,

$$\begin{aligned} \mathcal{O}_{\mathcal{X}^*}(U_i) &= \varprojlim_{\varepsilon} \mathcal{O}_{\mathcal{X}^*}(U_{i,\varepsilon}) = \varprojlim_{\varepsilon} (\mathcal{O}_{\mathcal{X}^*}(U_{i,\varepsilon})) \hat{\otimes}_{A_\varepsilon} K_r \\ &= \varprojlim_{\varepsilon} \mathcal{O}_{\mathcal{X}_t^\triangleright}(U_{i,\varepsilon} \cap \mathcal{X}_t^\triangleright) \\ &= \mathcal{O}_{\mathcal{X}_t^\triangleright}(U_i \cap \mathcal{X}_t^\triangleright), \end{aligned}$$

where the second equality is given by Lemma 4.4. Since the U_i form a G -cover of U , it follows that we have an isomorphism of k -algebras

$$\mathcal{O}_{T\mathcal{X} \setminus V(t)}(U) \cong \mathcal{O}_{\mathcal{X}^*}(\pi^{-1}(U)) \cong \mathcal{O}_{\mathcal{X}_t^\triangleright}(\pi^{-1}(U) \cap \mathcal{X}_t^\triangleright).$$

Moreover, we have also $\mathcal{O}_{T\mathcal{X} \setminus V(t)}^\circ(U) \cong \mathcal{O}_{\mathcal{X}_t^\triangleright}^\circ(\pi^{-1}(U) \cap \mathcal{X}_t^\triangleright)$, because, as noted in 3.9, an analytic function is bounded by 1 at a point x of $\mathcal{X}_t^\triangleright$ if and only if it is bounded by 1 at all the points of the orbit $\mathbb{R}_{>0} \cdot x$. It remains to show that φ is a homeomorphism of G -sites, i.e. that whenever U is an analytic domain of $\mathcal{X}_t^\triangleright$ then $\varphi(U)$ is an analytic domain of $T\mathcal{X}$. If V is an affinoid domain of $\mathcal{X}_t^\triangleright$, then it is also an affinoid domain of \mathcal{X}^* , hence the same reasoning as in the proof of 4.2 shows that $\pi^{-1}(\varphi(U)) = \mathbb{R}_{>0} \cdot U$ is an analytic domain of \mathcal{X}^* , therefore $\varphi(U)$ is an analytic domain of $T\mathcal{X}$. \square

(4.10) Corollary. *Let \mathcal{X} be a special formal scheme over k . Then the normalized Berkovich space $T\mathcal{X}$ of \mathcal{X} is locally isomorphic in the category \mathcal{C} to an object in the image of for $: (An_k((t))) \rightarrow \mathcal{C}$.*

Proof. Without loss of generality we can suppose that $\mathcal{X} = \mathrm{Spf}(A)$ is affine. Choose generators (f_1, \dots, f_s) for an ideal of definition of \mathcal{X} . Each f_i is topologically nilpotent in A and so induces a morphism $\mathcal{X} \rightarrow \mathrm{Spf}(k[[t]])$. Then the $\mathcal{X}^\triangleright \setminus V(f_i)$ cover \mathcal{X}^* , hence $T\mathcal{X}$ is covered by the $(\mathcal{X}^\triangleright \setminus V(f_i))/(\mathbb{R}_{>0})$ and by 4.9 $(\mathcal{X}^\triangleright \setminus V(f_i))/(\mathbb{R}_{>0}) \cong T\mathcal{X} \setminus V(f_i) \cong \mathrm{for}(\mathcal{X}_{f_i}^\triangleright)$. \square

(4.11) Remarks. If \mathcal{X}_t is a formal scheme of finite type over $k[[t]]$ then t does not vanish on $T\mathcal{X}$, and so Theorem 4.9 identifies $T\mathcal{X}$ with the image in \mathcal{C} of the $k((t))$ -analytic space $\mathcal{X}_t^\triangleright$. Note that the local $k((t))$ -analytic structures that we obtain in Corollary 4.10 are far from being unique. Indeed, not only we have to make choices of $|t| \in]0, 1[$, but we have to choose an affine cover of \mathcal{X} and generators of the ideals of definition of the elements of this covers. Nonetheless, for the sake of simplicity we will often refer to this result by saying that normalized k -spaces are locally $k((t))$ -analytic spaces.

(4.12) Example. If \mathcal{X} is the formal scheme $\mathrm{Spf}(\mathbb{C}[[X, Y]])$, then its normalized Berkovich space $T_{\mathcal{X}}$ is the valuative tree of [FJ04], as observed in 3.13. The largest ideal of definition of $\mathbb{C}[[X, Y]]$ is (X, Y) , so $T_{\mathcal{X}}$ is the union of the two $k((t))$ -analytic curves $\mathcal{X}_X^{\triangleright}$ and $\mathcal{X}_Y^{\triangleright}$, both isomorphic to the 1-dimensional open analytic disc over $k((t))$. It's important to remark that on their intersection, which is $T_{\mathcal{X}} \setminus V(XY)$, the two $k((t))$ -analytic structures do not agree. Actually, more is true: we will show in 4.16 that there is no $k((t))$ -analytic space C such that $T_{\mathcal{X}} \cong \mathrm{for}(C)$.

(4.13) Corollary 4.10 gives us a way of proving some assertions about analytic spaces over trivially valued fields by reducing to the non-trivially valued case. We prove in this way the analogue for normalized spaces of a result of A.J. de Jong [dJ95, 7.3.6]. De Jong shows that if $\mathcal{X} = \mathrm{Spf} A$ is a normal and affine flat special formal scheme over a complete discrete valuation ring R , then the formal functions on \mathcal{X} coincide with the bounded by 1 analytic functions on $\mathcal{X}^{\triangleright}$. More precisely, his proof applies under weaker assumptions: it is enough for A to be R -flat and integrally closed in the ring $A \otimes_R \mathrm{Frac}(R)$ (this is Remark 7.4.2 of [dJ95]). We can deduce that the same holds in our setting:

(4.14) Corollary. *Let A be a special k -algebra and assume that A is admissible and normal. If we denote by \mathcal{X} the formal scheme $\mathrm{Spf} A$, then the canonical morphism $A \rightarrow \Gamma(T_{\mathcal{X}}, \mathcal{O}_{T_{\mathcal{X}}})$ is an isomorphism.*

Proof. Since A is admissible, its largest ideal of definition is nonempty. If t is a nonzero element of this ideal, then A has no t -torsion since it is a domain; therefore A is flat over $k[[t]]$. Moreover, since A is normal then it is integrally closed in $A[t^{-1}]$. By letting t range among the nonzero elements of the largest ideal of definition of A we get a cover $T_{\mathcal{X}} = \bigcup_t \mathcal{X}_t^{\triangleright}$ as in 4.10. Then de Jong's theorem applies to $\mathcal{X}_t = \mathrm{Spf} A$, seen as a special formal scheme over the discrete valuation ring $k[[t]]$, yielding $A \cong \mathcal{O}_{\mathcal{X}_t^{\triangleright}}^{\circ}(\mathcal{X}_t^{\triangleright}) \cong \mathcal{O}_{T_{\mathcal{X}}}^{\circ}(T_{\mathcal{X}} \setminus V(t))$, therefore $A \cong \mathcal{O}_{T_{\mathcal{X}}}^{\circ}(T_{\mathcal{X}})$ since $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}$ is a sheaf. \square

(4.15) If T is an object of \mathcal{C} , we define the sheaf $\mathcal{O}_T^{\circ\circ}$ on T as the subsheaf of \mathcal{O}_T° consisting of the sections which are non-invertible in every stalk. If $T = T_{\mathcal{X}}$ is the normalized space of a special formal scheme \mathcal{X} over k which is covered by the formal spectra of normal and admissible special k -algebras, then $\mathcal{O}_T^{\circ\circ}$ coincides with the sheaf $\mathcal{O}_{T_{\mathcal{X}}}^{\circ\circ}$ defined earlier. Moreover, in this case the largest ideal of definition of \mathcal{X} coincides with $(\mathrm{sp}_{\mathcal{X}})^* \mathcal{O}_{T_{\mathcal{X}}}^{\circ\circ}$. Indeed, the elements of the largest ideal of definition of \mathcal{X} are precisely the ones that are topologically nilpotent, and this property can be verified by looking at the absolute values at every point.

(4.16) Example (continued). We have discussed in 4.12 a cover of the valuative tree $T_{\mathcal{X}} = T_{\mathrm{Spf}(\mathbb{C}[[X, Y]])}$ by $k((t))$ -analytic curves. Using Corollary 4.14 we now show that there is no global $k((t))$ -analytic structure on $T_{\mathcal{X}}$. To see this, assume that $T_{\mathcal{X}} \cong \mathrm{for}(C)$ for some $k((t))$ -analytic space C . Then the image of t in $\mathcal{O}_C(C) \cong \mathcal{O}_{T_{\mathcal{X}}}(T_{\mathcal{X}})$, which by abuse of notation we denote

again by t , has to be a nowhere vanishing function which is strictly bounded by 1, hence in particular an element of $\mathcal{O}_{T_{\mathcal{X}}}^{\circ\circ}(T_{\mathcal{X}})$. Since $\mathcal{O}_{T_{\mathcal{X}}}^{\circ\circ}(T_{\mathcal{X}})$ coincides with the largest ideal of definition (X, Y) of $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}(T_{\mathcal{X}}) \cong \mathbb{C}[[X, Y]]$, t is a complex power series in X and Y with no constant term and therefore defines the germ of a curve at the origin of $\mathbb{A}_{\mathbb{C}}^2$. Then the order of vanishing at the origin along this germ (which is a *curve valuation* in the terminology of [FJ04, 1.5.5]) defines a point of $T_{\mathcal{X}}$ on which t vanishes, giving a contradiction.

(4.17) Remark. There are other examples of subspaces of analytic spaces which are naturally analytic spaces locally but do not have a canonical field of definition. This is the case for the analytic boundaries of affinoid domains. This kind of behavior appears for example in [Duc12, Lemme 3.1].

We conclude the section by applying the results we have obtained to the study of admissible formal k -schemes. We start with an easy lemma.

(4.18) Lemma. *Let A be a special k -algebra. Then the morphism of formal schemes $\mathrm{Spf}(A/\mathrm{Tor}_A) \rightarrow \mathrm{Spf} A$ induced by the quotient $\pi : A \rightarrow A/\mathrm{Tor}_A$ gives an isomorphism on the level of normalized Berkovich spaces.*

Proof. Let $\{g_1, \dots, g_s\}$ be a set of generators of an ideal of definition of A and denote by \mathcal{X} and \mathcal{X}' the formal spectra of A and A/Tor_A respectively. Then $T_{\mathcal{X}}$ is covered by the Berkovich spaces $\mathcal{X}_{g_i}^{\triangleright}$, $T_{\mathcal{X}'}$ is covered by the $(\mathcal{X}')_{g_i}^{\triangleright}$ and the morphism $f : T_{\mathcal{X}'} \rightarrow T_{\mathcal{X}}$ induced by π is locally the morphism of Berkovich spaces induced by the morphism of special formal $k[[t]]$ -schemes $(\mathcal{X}')_{g_i} \rightarrow \mathcal{X}_{g_i}$ coming from π . The latter is an isomorphism since every element of Tor_A is g_i -torsion, hence f is an isomorphism. \square

We deduce the following result, which in turn implies that a formal $k[[t]]$ -scheme of finite type is an admissible special formal k -scheme if and only if it is admissible in the classical sense.

(4.19) Proposition. *Let $\mathcal{X} = \mathrm{Spf} A$ be an affine special formal scheme over k . Then \mathcal{X} is admissible if and only if the special k -algebra A is admissible.*

Proof. To prove the “if” part, since the topology on \mathcal{X} is generated by affine open formal subschemes, it is enough to show that the map $\varphi : A \rightarrow \mathcal{O}_{\mathcal{X}^*}(\mathcal{X}^*)$ is injective whenever A is admissible; 2.27 will then allow to conclude. So let a be an element of $A \setminus \{0\}$ such that $\varphi(a) = 0$. Choose t in A such that $a \notin \mathrm{Tor}_t(A)$ and consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A \otimes_{k[[t]]} k((t)) \\ \varphi \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}^*}(\mathcal{X}^*) & \longrightarrow & \mathcal{O}_{\mathcal{X}_t^{\triangleright}}(\mathcal{X}_t^{\triangleright}) \end{array}$$

where the bottom map is the restriction map under the identification of $\mathcal{X}_t^{\triangleright}$ with the subspace $\pi(\mathcal{X}^* \setminus V(t))$ of $T_{\mathcal{X}}$. The right vertical arrow is injective because t is invertible in $\mathcal{O}_{\mathcal{X}_t^{\triangleright}}(\mathcal{X}_t^{\triangleright})$. Since $a \notin \mathrm{Tor}_t(A)$ then a is sent to

a nonzero element by the top map, hence also $\varphi(a)$ is different from zero, which is what we had to prove. Now, to prove the “only if” part, denote by π the quotient map $A \rightarrow A/\mathrm{Tor}_A$, by \mathcal{X}' the formal spectrum of A/Tor_A , and consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/\mathrm{Tor}_A \\ \varphi \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}^*}(\mathcal{X}^*) & \xrightarrow{\cong} & \mathcal{O}_{\mathcal{X}'^*}(\mathcal{X}'^*) \end{array}$$

where the bottom arrow is induced by the morphism induced on formal spectra by π ; it is an isomorphism thanks to 4.18. It follows that φ is injective only if π is injective, that is only if $\mathrm{Tor}_A = 0$. \square

(4.20) Let \mathcal{X} be a special formal scheme over k , and let $\mathcal{T}_{\mathcal{X}}$ be the subsheaf of $\mathcal{O}_{\mathcal{X}}$ such that $\mathcal{T}_{\mathcal{X}}(\mathrm{Spf} A) = \mathrm{Tor}_A$ for every affine subscheme $\mathrm{Spf} A$ of \mathcal{X} . It's a coherent ideal subsheaf of $\mathcal{O}_{\mathcal{X}}$, so we can consider the quotient $\mathcal{O}_{\mathcal{X}}/\mathcal{T}_{\mathcal{X}}$. The special formal scheme $\mathcal{X}_{\mathrm{adm}}$ is defined as the closed formal subscheme of \mathcal{X} defined by $\mathcal{T}_{\mathcal{X}}$. It is an admissible special formal scheme over k which we call the *admissible special formal scheme associated to \mathcal{X}* . Its normalized Berkovich space coincides with the one of \mathcal{X} .

The following proposition is the analogue of an important classical result for formal R -schemes of finite type.

(4.21) Proposition. *Let \mathcal{X} be an admissible special formal k -scheme. Then the morphisms $\mathrm{sp}_{\mathcal{X}}: \mathcal{X}^* \rightarrow \mathcal{X}$ and $\mathrm{sp}_{\mathcal{X}}: T_{\mathcal{X}} \rightarrow \mathcal{X}$ are surjective.*

Proof. We can replace \mathcal{X} by its formal blowup along \mathcal{X}_0 , since the blowup morphism induces a surjective map between the reductions. Indeed, without loss of generality we can assume that $\mathcal{X} = \mathrm{Spf}(A)$ is affine and since it is admissible then no component of $\mathrm{Spec}(A)$ is contained in \mathcal{X}_0 , and therefore the image of the scheme theoretic blowup of $\mathrm{Spec}(A)$ along \mathcal{X}_0 , which is closed since the blowup is a proper map and has to contain the complement $\mathrm{Spec}(A) \setminus \mathcal{X}_0$, is all of $\mathrm{Spec}(A)$. Note that the blown up formal scheme is still admissible, because its ideal of definition is locally principal, generated by a regular element. By replacing \mathcal{X} with an affine open formal subscheme whose ideal of definition is principal, generated by an element t , we can assume that $\mathcal{X}_t = \mathcal{X} = \mathrm{Spf}(A)$ is an affine formal scheme of finite type over $k[[t]]$. Furthermore, we can assume that A is integrally closed in $A[t^{-1}]$, since the morphism of formal schemes induced by taking the integral closure induces a surjection at the level of special fibers. By [dJ95, 7.4.2] we have $A \cong \mathcal{O}_{\mathcal{X}_t^{\triangleright}}^{\circ}(\mathcal{X}_t^{\triangleright})$, therefore $(\mathcal{X}_t)_0 = \mathcal{X}_0$ is the canonical reduction of the affinoid space $\mathcal{X}_t^{\triangleright}$, and the map $\mathrm{sp}_{\mathcal{X}_t}: \mathcal{X}_t^{\triangleright} \rightarrow \mathcal{X}_0$ coincides with the reduction map of [Ber90, §2.4]. Therefore $\mathrm{sp}_{\mathcal{X}_t}$ is surjective by [Ber90, 2.4.4(i)]. The surjectivity of $\mathrm{sp}_{\mathcal{X}}: \mathcal{X}^* \rightarrow \mathcal{X}$ follows from the fact that $\iota \circ \mathrm{sp}_{\mathcal{X}} = \mathrm{sp}_{\mathcal{X}_t}$, where $\iota: \mathcal{X}_t^{\triangleright} \rightarrow \mathcal{X}^*$ is the natural inclusion of Lemma 4.7. \square

5. AFFINOID DOMAINS AND ATLASES

In this section we develop more in depth the analogy between normalized spaces over k and analytic spaces over $k((t))$, by defining the class of affinoid domains of a normalized space and showing that they behave like the affinoid domains of analytic spaces. In particular, in Proposition 5.6 we show that the G -topology of a normalized space can be described in terms of its affinoid domains, and in Proposition 5.9 we prove that normalized spaces are G -covered by finitely many affinoid domains. Theorem 5.14 shows that the property of being an affinoid domain of a normalized space is intrinsic, not depending on the choice of a $k((t))$ -analytic structure.

(5.1) Let V be an object of \mathcal{C} . We say that V is *affinoid* if it is isomorphic to $\text{for}(X)$ for some strictly affinoid $k((t))$ -analytic space X . An affinoid G -open V of an object T of \mathcal{C} is said to be an *affinoid domain* of T .

(5.2) Equivalently, V is affinoid if and only if it is isomorphic to the normalized space $T_{\mathcal{X}}$ of some affine formal $k[[t]]$ -scheme of finite type \mathcal{X}_t , since in this case $T_{\mathcal{X}} \cong \text{for}(\mathcal{X}_t^{\triangleright})$. If X is a strict $k((t))$ -affinoid, then as a $k((t))$ -analytic space we have $X \cong \mathcal{M}(\mathcal{O}_X^{\circ}(X) \hat{\otimes}_{k[[t]]} k((t))) \cong \text{Spf}(\mathcal{O}_X^{\circ}(X))_t^{\triangleright}$, and if moreover $\text{for}(X)$ is isomorphic to an object V of \mathcal{C} then $\mathcal{O}_X^{\circ}(X) \cong \mathcal{O}_V^{\circ}(V)$ as k -algebras. Therefore, V is affinoid if and only if the k -algebra $\mathcal{O}_V^{\circ}(V)$ can be endowed with a structure of $k[[t]]$ -algebra topologically of finite type and V is isomorphic to $\text{for}(\text{Spf}(\mathcal{O}_V^{\circ}(V))_t^{\triangleright}) \cong T_{\text{Spf}(\mathcal{O}_V^{\circ}(V))}$ in \mathcal{C} . If we want to remember the element $t \in \mathcal{O}_V^{\circ}(V)$ which is the image of t under the morphism $k[[t]] \rightarrow \mathcal{O}_V^{\circ}(V)$ making $\mathcal{O}_V^{\circ}(V)$ into a $k[[t]]$ -algebra topologically of finite type, we say that V is *affinoid with respect to the parameter t* , and by abuse of notation we will denote by V_t both the $k((t))$ -analytic space whose image in \mathcal{C} is isomorphic to V and the affinoid V itself. Observe that the parameter t is actually an element of $\mathcal{O}_V^{\circ\circ}(V)$.

(5.3) If an object V of \mathcal{C} is affinoid with respect to the parameter $t \in \mathcal{O}_V^{\circ\circ}(V)$, we say that V is a *principal affinoid with respect to t* if t generates $\mathcal{O}_V^{\circ\circ}(V)$.

(5.4) Equivalently, V is a principal affinoid domain if it is isomorphic to $\text{for}(X)$ where X is a strict $k((t))$ -affinoid space which is *principal*, by which we mean that the special fiber \mathcal{X}_s of its canonical formal model $\mathcal{X} = \text{Spf } \mathcal{O}_X^{\circ}(X)$ is reduced. Indeed, the canonical model $\mathcal{X} = \text{Spf } \mathcal{O}_X^{\circ}(X)$ of X has special fiber $\mathcal{X}_s = \text{Spec}(\mathcal{O}_X^{\circ}(X) \otimes_{k[[t]]} k)$, and $(\mathcal{X}_s)_{\text{red}} = \mathcal{X}_0 = \text{Spec}(\mathcal{O}_X^{\circ}(X)/\mathcal{O}_X^{\circ\circ}(X))$, so \mathcal{X}_s is reduced if and only if $\mathcal{O}_X^{\circ\circ}(X)$ is generated by t .

(5.5) **Remark.** Some properties of a strict $k((t))$ -affinoid V depend only on its normalized space structure, for example its canonical reduction, which is the k -scheme $\text{Spec}(\mathcal{O}_V^{\circ}(V)/\mathcal{O}_V^{\circ\circ}(V))$, and its canonical formal model, which is $\text{Spf}(\mathcal{O}_V^{\circ}(V))$, when seen as a formal k -scheme. However, if V is an object of \mathcal{C} which is affinoid with respect to two different parameters t_1 and t_2 , it is not true in general that V_{t_1} and V_{t_2} are isomorphic as $k((t))$ -analytic spaces. For example, the $k((t))$ -analytic spaces $\mathcal{M}(k((t)))$ and $\mathcal{M}(k((t))\{X\}/(X^2 - t))$

are not isomorphic since $k((t))$ and $k((t))\{X\}/(X^2 - t)$ are not isomorphic as $k((t))$ -algebras, but the associated normalized spaces coincide because $k((t))$ and $k((t))\{X\}/(X^2 - t)$ are isomorphic as special k -algebras.

The following proposition pushes further the analogies between usual Berkovich spaces and normalized spaces, showing that the G -topology of a normalized space can be described in terms of its affinoid domains.

(5.6) Proposition. *Let \mathcal{X} be a special formal scheme over k and let U be a subspace of the normalized space $T_{\mathcal{X}}$ of \mathcal{X} . If U is an analytic domain of $T_{\mathcal{X}}$, then it is G -covered by affinoid domains of $T_{\mathcal{X}}$.*

Proof. Since $T_{\mathcal{X}}$ is G -covered by the normalized spaces of the affine open formal subschemes of \mathcal{X} , we can assume that \mathcal{X} is itself affine. Assume that U is an analytic domain of $T_{\mathcal{X}}$, i.e. that $\pi^{-1}(U)$ is an analytic domain of \mathcal{X}^* . Cover $T_{\mathcal{X}}$ by the $k((t))$ -analytic spaces $\mathcal{X}_{t_i}^{\supseteq}$, for t_i ranging over a finite set of generators of an ideal of definition of \mathcal{X} , and set $U_i = \pi^{-1}(U) \cap \mathcal{X}_{t_i}^{\supseteq}$. Then the U_i are analytic domains of $\mathcal{X}_{t_i}^{\supseteq}$, so they are G -covered by affinoid domains $V_{i,j}$ of $\mathcal{X}_{t_i}^{\supseteq}$, and the $\pi(V_{i,j})$ are affinoid domains of $T_{\mathcal{X}}$. Finally, the $\pi(V_{i,j})$ form a G -cover of U , since the $V_{i,j}$, and therefore also the $\pi^{-1}\pi(V_{i,j})$, form a G -cover of $\pi^{-1}(U)$. \square

(5.7) Remark. Proposition 5.6 tells us that we can think about the G -topology of $T_{\mathcal{X}}$ the same way we think about the one of a Berkovich space. If U is a subspace of $T_{\mathcal{X}}$, then U is an analytic domain if and only if it is covered by a family $\{V_i\}_{i \in I}$ of affinoid domains and the following property holds: for every point x of U , there exists a finite subset I_x of I such that $\bigcup_{i \in I_x} V_i$ contains an open neighborhood of x and $x \in \bigcap_{i \in I_x} V_i$.

(5.8) We showed in 4.10 that the normalized Berkovich space $T_{\mathcal{X}}$ of a special formal k -scheme \mathcal{X} is locally a $k((t))$ -analytic space. We will now describe a second way of covering $T_{\mathcal{X}}$ by $k((t))$ -analytic spaces which will be very useful later; the price to pay is that we are obliged to change the formal scheme \mathcal{X} . If T is an object of \mathcal{C} , we call *atlas* of T a G -cover of T by affinoid domains.

(5.9) Proposition. *Let \mathcal{X} be a special formal k -scheme. Then the normalized Berkovich space $T_{\mathcal{X}}$ of \mathcal{X} admits a finite atlas.*

Proof. By replacing \mathcal{X} with the associated admissible special formal k -scheme, as defined in 4.20, we can assume that \mathcal{X} is admissible. Performing an admissible blowup $\mathcal{X}' \rightarrow \mathcal{X}$ we can assume that the largest ideal of definition of \mathcal{X}' is locally principal. Now, by locally sending t to a generator of this ideal, we cover \mathcal{X}' by finitely many affine open formal schemes of finite type over $k[[t]]$. Their normalized spaces then form an atlas of $T_{\mathcal{X}'} \cong T_{\mathcal{X}}$. \square

(5.10) Example. In the case of the valuative tree of 3.13, we get an atlas by blowing up the origin of $\mathbb{A}_{\mathbb{C}}^2$ and using the two charts of the blowup.

In the remaining of the section we give a criterion for an object of \mathcal{C} to be affinoid which doesn't require checking the existence of a parameter t , following the approach of [Liu90] for rigid analytic spaces.

(5.11) Let X be a locally ringed G -topological space. Following [Liu90], we say that X is a *Stein space* if we have $H^n(X, \mathcal{F}) = 0$ for every coherent sheaf of \mathcal{O}_X -modules \mathcal{F} and every $n \geq 1$.

Let T be an object of \mathcal{C} . We say that T is *compact* if every G -cover of T has a finite G -subcover. Finally, we say that T is *pseudo-affinoid* if it is isomorphic to $\text{for}(\mathcal{X}_t^\triangleright)$ for some affine special formal scheme \mathcal{X}_t over $k[[t]]$.

(5.12) Proposition. *Every pseudo-affinoid object of \mathcal{C} is a Stein space. If X is a $k((t))$ -analytic space, then X is a $k((t))$ -affinoid space if and only if $\text{for}(X)$ is both compact and pseudo-affinoid.*

Proof. If an object T of \mathcal{C} is pseudo-affinoid then as discussed in 2.17 it is the increasing union of the affinoid domains W_ε , and we have surjective restriction morphisms $\mathcal{O}_T(W_\varepsilon) \rightarrow \mathcal{O}_T(W_{\varepsilon'})$ when $\varepsilon > \varepsilon'$, so T is quasi-Stein in the sense of Kiehl. It follows that pseudo-affinoids are Stein spaces since Kiehl's Theorem B [Kie67, 2.4] applies (see also [Nic09, §2.1] for a definition of quasi-Stein and the statement of Kiehl's theorem). To prove the second claim, note that a $k((t))$ -affinoid space X is compact, so $\text{for}(X)$ is compact as well, and it is clearly pseudo-affinoid. Conversely, if $\text{for}(X)$ is pseudo-affinoid then the W_ε above form a G -covering of it and so, since this family is increasing, by compactness $\text{for}(X)$ coincides with one of the W_ε , which is an affinoid domain with respect to the parameter t . Therefore X is itself affinoid. \square

(5.13) Liu has proven in [Liu90, 3.2] that if X and Y are two rigid spaces over a non-archimedean field K and X is Stein and quasi-compact, then the canonical map $\text{Hom}_{K\text{-an}}(Y, X) \rightarrow \text{Hom}_{K\text{-alg}}(\mathcal{O}_X(X), \mathcal{O}_Y(Y))$ is a bijection. He deduced that a rigid space X over K is strictly affinoid if and only if it is Stein, $\mathcal{O}_X(X)$ is a strict affinoid K -algebra and X is quasi-compact [Liu90, 3.2.1]. We give now an analogous result for the objects of the category \mathcal{C} .

(5.14) Theorem. *Let X be a $k((t))$ -analytic space. Then X is strictly $k((t))$ -affinoid if and only if $\text{for}(X)$ is Stein and compact and $\mathcal{O}_X^\circ(X)$, with its $\mathcal{O}_X^\circ(X)$ -adic topology, is a special k -algebra.*

Proof. If A is a strictly affinoid $k((t))$ -algebra then A° is a special k -algebra, so the “only if” implication is clear. For the converse implication, define $\mathcal{X} = \text{Spf}(\mathcal{O}_X^\circ(X))$. Since X is a $k((t))$ -analytic space, the image of t in $\mathcal{O}_X(X)$ is strictly bounded by 1, hence it's an element of the largest ideal of definition of $\mathcal{O}_\mathcal{X}(\mathcal{X})$, so \mathcal{X}_t is an affine special formal scheme over $k[[t]]$. It follows that $\mathcal{X}_t^\triangleright$ is a pseudo-affinoid space. Since X is compact and $|t| < 1$, we have $\mathcal{O}_X^\circ(X)[t^{-1}] \cong \mathcal{O}_X(X)$. Indeed, any f in $\mathcal{O}_X(X)$ is bounded on X , hence ft^n is bounded by 1 for n big enough. Therefore, by [Liu90, 3.2], as recalled in 5.13, the canonical homomorphism of $k((t))$ -algebras $\mathcal{O}_X^\circ(X)[t^{-1}] \rightarrow \mathcal{O}_{\mathcal{X}_t^\triangleright}(\mathcal{X}_t^\triangleright)$ induces a morphism of rigid $k((t))$ -spaces $\mathcal{X}_t^{\text{rig}} \rightarrow X^{\text{rig}}$. On rigid points, this morphism is obtained as follows: by [Liu90, 1.3] $X^{\text{rig}} = \text{Specmax}(\mathcal{O}_X(X))$, and a rigid point of $\mathcal{X}_t^\triangleright$, which corresponds by [dJ95, 7.1.9] to a maximal ideal of $\mathcal{O}_{\mathcal{X}_t^\triangleright}(\mathcal{X}_t^\triangleright)$, is sent to the

inverse image of this ideal through the composition $\mathcal{O}_X(X) \rightarrow \mathcal{O}_{\mathcal{X}_t^\triangleright}(\mathcal{X}_t^\triangleright)$. This morphism $X^{\text{rig}} \rightarrow \mathcal{X}_t^{\text{rig}}$ is then the bijection constructed by de Jong in *loc.cit.*, and by combining that result with [Liu90, 1.3] we see that it induces isomorphisms at the level of completed local rings. This implies that $\mathcal{X}_t^\triangleright \cong X$, since by [BGR84, 7.3.5] a bijective morphism between rigid analytic spaces which induces isomorphisms on completed local rings is an isomorphism, and by [Ber93] two analytic spaces are isomorphic if the underlying rigid spaces are isomorphic. Hence, being both pseudo-affinoid and compact, X is affinoid by 5.12. Since $\mathcal{O}_X^\circ(X)$ is a special k -algebra, X is moreover strict. \square

A $k((t))$ -analytic space X is Stein if and only if $\text{for}(X)$ is, because the groups $H^n(X, \mathcal{F})$ depend only on the locally ringed space (X, \mathcal{O}_X) and not on the $k((t))$ -algebra structure on \mathcal{O}_X . Since the same is true for the two other properties in the statement above, we obtain the following corollary.

(5.15) Corollary. *Let X and X' be $k((t))$ -analytic spaces such that $\text{for}(X) \cong \text{for}(X')$. Then X is strictly affinoid if and only if X' is strictly affinoid.*

If V is a subspace of an object of \mathcal{C} of the form $\text{for}(X)$ for some $k((t))$ -analytic space X , then V inherits the structure of a $k((t))$ -analytic space. Therefore, the last corollary has the following useful consequence.

(5.16) Corollary. *Let X be a $k((t))$ -analytic space and let V be a subspace of $\text{for}(X)$. Then V is affinoid if and only if it is of the form $\text{for}(W)$ for some strict affinoid domain W of X . Moreover, V is principal affinoid with respect to $t|_V$ if and only if W is a principal strictly affinoid $k((t))$ -analytic space.*

6. FUNCTORIALITY

In this section we introduce the category of normalized spaces over k . The main result, Theorem 6.4, is the analogue for normalized spaces of the fundamental result of Raynaud ([Ray74], a detailed proof is [BL93, 4.1]) which states that the functor $\mathcal{X} \mapsto \mathcal{X}^\triangleright$ induces an equivalence between the category of admissible formal schemes of finite type over a (non-trivially valued) valuation ring R , localized by the class of admissible blowups, and the category of quasi-compact and quasi-separated analytic spaces over $\text{Frac}(R)$.

(6.1) We say that an object T of \mathcal{C} is *quasi-separated* if the intersection of any two affinoid domains of T is a finite union of affinoid domains of T . We say that T is a *normalized space* if it is quasi-separated, compact and it has an atlas. A morphism $f \in \text{Hom}_{\mathcal{C}}(T', T)$ is said to be a *morphism of normalized spaces* if there exist finite covers $\{V_i\}_{i \in I}$ of T and $\{W_j\}_{j \in J}$ of T' by affinoid domains such that for every i in I there is a subset J_i of J so that $f^{-1}(V_i) = \cup_{j \in J_i} W_j$ and, for every j in J_i , the restriction $f|_{W_j} : W_j \rightarrow V_i$ of f to W_j is induced by a morphism of $k((t))$ -analytic spaces. The *category of normalized spaces* (NAn_k) is the subcategory of \mathcal{C} whose objects are normalized spaces and whose morphisms are morphisms of normalized spaces.

(6.2) We want to prove a result analogous to Raynaud's theorem for the functor $T : \mathcal{X} \mapsto T_{\mathcal{X}}$. The source category will be the category of admissible special formal k -schemes with adic morphisms, which we denote by $(SFor_k)$. We will show that the functor T is the localization of the category $(SFor_k)$ by the class B of admissible formal blowups and we will characterize its essential image. The fact that T sends admissible blowups to isomorphisms is 2.24.

(6.3) The category $(SFor_k)$ admits calculus of (right) fractions with respect to the class of morphisms B , in the sense of [GZ67, Ch. I]. This follows easily from the universal property of blowing-up and the results of 2.9. Therefore, the localized category $(SFor_k)_B$ can be described in a simple way: its objects are the objects of $(SFor_k)$, and a morphism $\mathcal{Y} \rightarrow \mathcal{X}$ is a two-step zigzag

$$\mathcal{Y} \xleftarrow{w} \mathcal{Y}' \xrightarrow{f} \mathcal{X}$$

where f is a morphism in $(SFor_k)$ and w is an admissible blowup, modulo the equivalence relation given by further blowing up \mathcal{Y}' . Such a morphism can be thought of as a fraction fw^{-1} . Moreover, the localization functor $(SFor_k) \rightarrow (SFor_k)_B$ is left exact, and therefore preserves finite limits.

(6.4) Theorem. *The functor $T : \mathcal{X} \mapsto T_{\mathcal{X}}$ induces an equivalence between the category $(SFor_k)_B$, the localization of the category $(SFor_k)$ of admissible special formal k -schemes with adic morphisms by the class B of admissible blowups, and the category (NAn_k) .*

The rest of this section will be devoted to the proof of this result.

(6.5) Lemma. *Let $T_{\mathcal{X}}$ be the normalized space of an admissible special formal k -scheme \mathcal{X} . Then:*

- (i) *If V is an affinoid domain of $T_{\mathcal{X}}$, then there exist an admissible formal blowup $\mathcal{X}' \rightarrow \mathcal{X}$, an affine formal $k[[t]]$ -scheme of finite type \mathcal{V} and an open immersion of formal k -schemes $\mathcal{V} \hookrightarrow \mathcal{X}'$ inducing an isomorphism $T_{\mathcal{V}} \cong V$ in \mathcal{C} .*
- (ii) *If $\{V_j\}_{j \in J}$ is a finite atlas of $T_{\mathcal{X}}$, then there exist an admissible formal blowup $\mathcal{X}' \rightarrow \mathcal{X}$ and a cover $\{\mathcal{V}_j\}_{j \in J}$ of \mathcal{X}' by open formal subschemes such that $T_{\mathcal{V}_j} \cong V_j$ for every j .*

Proof. We prove (i) by reducing to the classical case of formal $k[[t]]$ -schemes of finite type, where it is [Bos14, §2.8, Lemma 4]. After an admissible blowup we can assume that \mathcal{X} is covered by open formal $k[[t]]$ -schemes of finite type \mathcal{X}_i , as in the proof of 5.9. Then $T_{\mathcal{X}} = \bigcup_i T_{\mathcal{X}_i}$, and so $V = \bigcup_i (V \cap T_{\mathcal{X}_i})$. Inducting on i , [Bos14, §2.8, Lemma 4] tells us that after an admissible blowup \mathcal{X}'_i of \mathcal{X}_i , which extends to an admissible blowup \mathcal{X}' of \mathcal{X} , we can find an open formal $k[[t]]$ -subscheme \mathcal{V}_i of \mathcal{X}'_i , which is also an open formal k -subscheme of \mathcal{X}' , such that $T_{\mathcal{V}_i} = V_i$. We obtain an open formal k -subscheme $\mathcal{V} = \bigcup_i \mathcal{V}_i$ of an admissible blowup of \mathcal{X} such that $T_{\mathcal{V}} = V$, which is what we wanted. To prove (ii), we apply (i) to get for every j an admissible blowup

\mathcal{X}_j of \mathcal{X} with an open formal subscheme $\mathcal{W}_j \subset \mathcal{X}_j$ such that $T_{\mathcal{W}_j} \cong V_j$. Using 2.9.ii we take an admissible blowup \mathcal{X}' of \mathcal{X} dominating all \mathcal{X}_j via maps $f_j : \mathcal{X}' \rightarrow \mathcal{X}_j$, and we set $\mathcal{V}_j := f_j^{-1}(\mathcal{W}_j)$. The \mathcal{V}_j form a cover of \mathcal{X}' which satisfies the requirements since $T_{\mathcal{V}_j} \cong T_{\mathcal{W}_j} \cong V_j$. \square

The rest of the proof of Theorem 6.4 will be divided in six steps. While the result could be proven in an analogous way as Raynaud did, we decided to deduce it from his result, to give an idea of how to apply to normalized spaces standard techniques over $k[[t]]$, in a similar way as what we did in 6.5.

Step 1: The functor T factors through (NAn_k) . If \mathcal{X} is a special formal scheme over k , then $T_{\mathcal{X}}$ is a normalized space, since it admits a finite atlas by Proposition 5.9, and quasi-separatedness follows from Lemma 6.5: if V and W are affinoid domains of $T_{\mathcal{X}}$, then $V \cap W \cong T_{V \cap W}$ for some open formal k -subscheme \mathcal{V} and \mathcal{W} of an admissible blowup \mathcal{X}' of \mathcal{X} , so $V \cap W$ has itself a finite atlas. Let now $f : \mathcal{Y} \rightarrow \mathcal{X}$ be an adic morphism of special formal k -schemes. By replacing \mathcal{X} and \mathcal{Y} by admissible blowups we obtain an adic morphism $f' : \mathcal{Y}' \rightarrow \mathcal{X}'$ and we can assume that \mathcal{X}' is covered by affine open formal subschemes \mathcal{X}_i of finite type over $k[[t]]$. The open formal k -subscheme $(f')^{-1}(\mathcal{X}_i)$ of \mathcal{Y}' are themselves covered by affine open formal k -subscheme $\mathcal{Y}_{i,j}$, and since f' is adic, the morphisms of formal k -schemes $f'|_{\mathcal{Y}_{i,j}} : \mathcal{Y}_{i,j} \rightarrow \mathcal{X}_i$ can be upgraded to a morphism of formal $k[[t]]$ -schemes, by choosing the $k[[t]]$ -structure on $\mathcal{Y}_{i,j}$ given by the parameter $f'_{\#}t$, and therefore it induces a morphism of $k((t))$ -analytic spaces $(\mathcal{Y}_{i,j})_t^{\triangleright} \rightarrow (\mathcal{X}_i)_t^{\triangleright}$. This shows that T_f is a morphism in (NAn_k) .

Step 2: Faithfulness. Let $f, g : \mathcal{Y} \rightarrow \mathcal{X}$ be two morphisms in $(SFor_k)$ such that the induced morphisms of normalized spaces $f_T, g_T : T_{\mathcal{Y}} \rightarrow T_{\mathcal{X}}$ coincide. Since \mathcal{X} and \mathcal{Y} are admissible, the specialization maps $T_{\mathcal{X}} \rightarrow \mathcal{X}$ and $T_{\mathcal{Y}} \rightarrow \mathcal{Y}$ are surjective by 4.21. Consider the following diagram:

$$\begin{array}{ccc} T_{\mathcal{Y}} & \xrightarrow{f_T=g_T} & T_{\mathcal{X}} \\ \text{sp}_{\mathcal{Y}} \downarrow & & \downarrow \text{sp}_{\mathcal{X}} \\ \mathcal{Y} & \xrightarrow[\quad g \quad]{\quad f \quad} & \mathcal{X} \end{array}$$

It commutes for both choices of the map on bottom, so it follows that f and g coincide as maps between the topological spaces underlying \mathcal{Y} and \mathcal{X} . Therefore we can assume that \mathcal{X} and \mathcal{Y} are affine, $\mathcal{X} = \text{Spf}(A)$ and $\mathcal{Y} = \text{Spf}(B)$, so that f and g correspond to two k -algebra maps $f_{\#}$ and $g_{\#} : A \rightarrow B$ respectively. Consider then the following diagram,

$$\begin{array}{ccc} A & \xrightarrow[\quad g_{\#} \quad]{\quad f_{\#} \quad} & B \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathcal{X}^*) & \xrightarrow{(f^*)_{\#}=(g^*)_{\#}} & \mathcal{O}(\mathcal{Y}^*) \end{array}$$

which commutes for both choices of the map on top. The vertical arrows are injective since \mathcal{X} and \mathcal{Y} are admissible, so $f_{\sharp} = g_{\sharp}$, and hence $f = g$.

Step 3: Fullness (modulo admissible blowup). Let $f : T' \rightarrow T$ be a morphism in (NAn_k) and let \mathcal{X} and \mathcal{Y} be models of T and T' respectively. Given two finite affine covers of \mathcal{X} and of \mathcal{Y} , using the fact that admissible blowups open formal subschemes can be extended by 2.9.iii, after blowing up \mathcal{Y} we can refine them to finite covers $\{(V_i)_t\}$ of \mathcal{X} and $\{(W_j)_t\}$ of \mathcal{Y} by affine formal schemes of finite type over $k[[t]]$ in such a way that, if we define $k((t))$ -analytic spaces $V_i = (\mathcal{V}_i)_t^{\rhd}$ and $W_j = (\mathcal{W}_j)_t^{\rhd}$, then $\{\text{for}(V_i)\}$ and $\{\text{for}(W_j)\}$ are covers of T and T' respectively as the ones in the definition of a morphism of normalized spaces. For every j we use Raynaud's theorem [BL85, 4.1]: after blowing up $(\mathcal{W}_j)_t$ to $(\mathcal{W}'_j)_t$, the analytic morphism $f|_{W_j} : W_j \rightarrow V_i$ lifts to a morphism $F : (\mathcal{W}'_j)_t \rightarrow (\mathcal{V}_i)_t$ of formal schemes of finite type over $k[[t]]$. These morphisms glue to a morphism $F : \mathcal{Y}' \rightarrow \mathcal{X}$ of formal schemes over k from a blowup \mathcal{Y}' of \mathcal{Y} to \mathcal{X} , and $T(F) = f$ since this is the case locally. The morphism F is adic since it is locally a morphism of $k[[t]]$ -formal schemes of finite type and such morphisms are always adic.

Step 4: Isomorphisms come from admissible blowups. If in the previous step we take for f an isomorphism in \mathcal{C} , then it can be lifted to an admissible blowup $F : \mathcal{X} \rightarrow \mathcal{Y}$. Indeed, if f is an isomorphism, then the analytic morphisms $f|_{W_j} : W_j \rightarrow V_i$ of the previous step are G -open immersions of $k((t))$ -analytic spaces, so we can use the analogous result in Raynaud theory.

Step 5: Existence of a model. Let T be an element of (NAn_k) and let $\{X_i\}_{i \in I}$ be a finite atlas of T . We will prove the existence of a model of T , i.e. a special formal k -scheme \mathcal{X} such that $T_{\mathcal{X}} \cong T$, by induction on the cardinality of I . If I consists of only one element then T is affinoid, and therefore $\text{Spf}(\mathcal{O}_T^\circ(T))$ is a model of T . If $I = \{1, 2, \dots, n\}$, X_n being affinoid has a model \mathcal{U} , and by induction we can find a model \mathcal{V} of $V := X_1 \cup \dots \cup X_{n-1}$. Set $W := V \cap X_n$. Since T is quasi-separated, W admits a finite cover by affinoid domains, and this cover can be enlarged to a cover of V by affinoid domains. By Lemma 6.5, there exist an admissible blowup $\mathcal{V}' \rightarrow \mathcal{V}$ and an open immersion $\mathcal{W}_1 \hookrightarrow \mathcal{V}'$ inducing an isomorphism $T_{\mathcal{W}_1} \cong W$. Similarly, there exist an admissible blowup $\mathcal{U}' \rightarrow \mathcal{U}$ and an open immersion $\mathcal{W}_2 \hookrightarrow \mathcal{U}'$ inducing an isomorphism $T_{\mathcal{W}_2} \cong W$. Since \mathcal{W}_1 and \mathcal{W}_2 are both models of W , using Step 4 we can find an admissible blowup \mathcal{W} of both \mathcal{W}_1 and \mathcal{W}_2 . These blowups can be extended using 2.9.iii to admissible blowups $\mathcal{V}'' \rightarrow \mathcal{V}$ and $\mathcal{U}'' \rightarrow \mathcal{U}$, so we can glue \mathcal{V}'' and \mathcal{U}'' along \mathcal{W} , obtaining a model of T .

Step 6: End of proof. It remains to prove that the functor T satisfies the universal property of the localization of categories. The fact that T sends admissible blowups to isomorphisms is 2.24. Given a category \mathcal{C} and a functor $F : (SFor_k) \rightarrow \mathcal{C}$ such that $F(b)$ is an isomorphism in \mathcal{C} for every admissible blowup b , we need to show that F factors as $G \circ T$ for a unique functor $G : (NAn_k) \rightarrow \mathcal{C}$. This is done in the exact same way as in the proof of Raynaud's theorem, using the basic results on blowups of 2.9.

This completes the proof of Theorem 6.4. \square

(6.6) Remark. In particular, an analytic space of the form \mathcal{X}^* for some special formal k -scheme \mathcal{X} is uniquely determined by the associated normalized space. An explicit way of retrieving the topological space underlying \mathcal{X}^* from $T_{\mathcal{X}}$ is the following. If we cover $T_{\mathcal{X}}$ by affinoid subspaces X_i with respect to parameters t_i , we need to glue the topological spaces $X_i \times \mathbb{R}_{>0}$ along subspaces homeomorphic to $X_{ij} \times \mathbb{R}_{>0}$. The gluing data is encoded in the t_i : if x is a point of X_{ij} , we identify $(x, \gamma) \in X_i \times \mathbb{R}_{>0}$ to $(x, \lambda_{ij}(x)\gamma) \in X_j \times \mathbb{R}_{>0}$, where $\lambda_{ij}(x) := \log |t_j| / \log |t_i|(x)$ is defined as in Remark 3.10.

7. MODIFICATIONS OF SURFACES AND VERTEX SETS

Starting from this section we move to the study of pairs (X, Z) , where X is a surface over an arbitrary (trivially valued) field k and Z is a closed subvariety of X containing its singular locus. After giving some general definitions, in Theorem 7.18 we use normalized spaces to produce formal modifications of (X, Z) with prescribed exceptional divisors.

(7.1) Let X be a surface over k and let $Z \subsetneq X$ be a nonempty closed subscheme whose support contains the singular locus of X . We denote by \mathcal{X} the formal completion of X along Z , and by $T_{X,Z}$ the normalized space $T_{\mathcal{X}}$ of \mathcal{X} . We also call $T_{X,Z}$ the *normalized space of the pair* (X, Z) . As discussed in 3.14, $T_{X,Z}$ can be viewed as a non-archimedean model for the link of Z in X . We denote by $\widetilde{\text{Bl}_Z X}$ the normalization of the blowup of X in Z , and we write $\widetilde{\mathcal{X}}$ for the formal completion of $\widetilde{\text{Bl}_Z X}$ along $\widetilde{\text{Bl}_Z X} \times_X Z$. Then $\widetilde{\mathcal{X}}$ is another special formal k -scheme which is a model of $T_{X,Z}$. By [Nic09, 2.16.5], $\widetilde{\mathcal{X}}$ is isomorphic to the normalization of the formal blowup of \mathcal{X} along Z .

(7.2) A *log modification* of the pair (X, Z) is a pair (Y, D) consisting of a normal k -variety Y and a Cartier divisor D of Y , together with a proper morphism of k -varieties $f : Y \rightarrow X$ such that $D = Y \times_X Z$ as subschemes of Y and f is an isomorphism out of D . A log modification (Y, D) of (X, Z) is said to be a *log resolution* of (X, Z) if Y is regular and D has normal crossings. Note that the condition we impose on D is not equivalent to the fact that the set-theoretic inverse image $f^{-1}(Z) = D_{\text{red}}$ of Z is a divisor with normal crossings of Y , therefore our notion of log resolution is different from the notion of good resolution which is sometimes found in the literature.

(7.3) A *formal log modification* of the pair (X, Z) is a normal special formal k -scheme \mathcal{Y} together with an adic morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ which induces an isomorphism of normalized spaces $T_{\mathcal{Y}} \xrightarrow{\sim} T_{X,Z}$, and such that $\mathcal{Y} \times_{\mathcal{X}} Z$ is a Cartier divisor of \mathcal{Y} . If moreover \mathcal{Y} is regular and $\mathcal{Y} \times_{\mathcal{X}} Z$ has normal crossings in \mathcal{Y} , \mathcal{Y} is said to be a *formal log resolution* of (X, Z) .

(7.4) Lemma. *Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a formal log modification of (X, Z) . Then f is proper.*

Proof. Since f is adic by definition, it is enough to show that the induced morphism $f_0 : \mathcal{Y}_0 \rightarrow \mathcal{X}_0$ is a proper morphism of schemes. Consider the morphism $\alpha : T_{\mathcal{X}} \rightarrow T_{\mathcal{Y}}$, inverse of the isomorphism induced by f . Theorem 6.4 states that there exists an admissible blowup $\tau : \mathcal{X}' \rightarrow \mathcal{X}$ and a morphism $g : \mathcal{X}' \rightarrow \mathcal{Y}$ such that α is induced by g . Since the composition $\mathcal{X}' \xrightarrow{g} \mathcal{Y} \xrightarrow{f} \mathcal{X}$ is τ , the induced map $f_0 \circ g_0 : \mathcal{X}'_0 \rightarrow \mathcal{X}_0$ is proper. The map g_0 is surjective because $\mathrm{sp}_{\mathcal{Y}}$ is surjective and the following diagram

$$\begin{array}{ccc} T_{\mathcal{X}'} & \xrightarrow{\simeq} & T_{\mathcal{Y}} \\ \mathrm{sp}_{\mathcal{X}'} \downarrow & & \downarrow \mathrm{sp}_{\mathcal{Y}} \\ \mathcal{X}'_0 & \xrightarrow{g_0} & \mathcal{Y}_0 \end{array}$$

is commutative. Since surjectivity is stable under base change by [Gro60, 3.5.2], it follows that f_0 is universally closed and therefore proper. \square

(7.5) If $(Y, D) \rightarrow (X, Z)$ is a log modification, then the formal completion $\mathcal{Y} = \widehat{Y/D} \rightarrow \mathcal{X}$ of Y along D is a formal log modification of (X, Z) . Such a formal log modification \mathcal{Y} of (X, Z) is said to be *algebraizable*, and a log modification (Y, D) of (X, Z) such that $\mathcal{Y} \rightarrow \mathcal{X}$ is isomorphic to $\widehat{Y/D} \rightarrow \mathcal{X}$ is called an *algebraization* of \mathcal{Y} . By [Gro61, 5.1.4] a log modification $(Y, D) \rightarrow (X, Z)$ is uniquely determined by the formal log modification $\widehat{Y/D} \rightarrow \mathcal{X}$ it algebraizes, and if $\mathcal{Y} \cong \widehat{Y/D}$ and $\mathcal{Y}' \cong \widehat{Y'/D'}$ are two formal log modifications that are algebraizable then $\mathrm{Hom}_{\mathcal{X}}(\mathcal{Y}', \mathcal{Y}) \cong \mathrm{Hom}_X(Y', Y)$. If \mathcal{Y} is a formal log modification of (X, Z) algebraized by the log modification (Y, D) , since both the properties of being regular and of having normal crossings can be checked on completed local rings, then \mathcal{Y} is a formal log resolution of (X, Z) if and only if (Y, D) is a log resolution of (X, Z) .

(7.6) Proposition. *Every formal log resolution of (X, Z) is algebraized by a log resolution of (X, Z) .*

Proof. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a formal log resolution of (X, Z) . The map f factors through a morphism $g : \mathcal{Y} \rightarrow \widetilde{\mathcal{X}}$. It follows from Lemma 7.4 that g makes \mathcal{Y} into a proper, adic formal $\widetilde{\mathcal{X}}$ -scheme. Since $\widetilde{\mathcal{X}}$ is normal, g is an isomorphism out of the inverse image of a finite set of closed points of $\widetilde{\mathcal{X}}$. Let \mathcal{U} be a formal subscheme of $\widetilde{\mathcal{X}}$ such that there is exactly one point x in \mathcal{U} such that $g|_{g^{-1}(\mathcal{U})}$ is an isomorphism out of $g^{-1}(x)$, and denote by E_1, \dots, E_r the irreducible components of $g^{-1}(x)$. Since $g^{-1}(\mathcal{U}) \subset \mathcal{Y}$ is regular, each E_i is a Cartier divisor on $g^{-1}(\mathcal{U})$. The intersection matrix $(E_i \cdot E_j)_{1 \leq i, j \leq r}$ is negative definite, so we can find a Cartier divisor $E = \sum_i a_i E_i$ such that $E \cdot E_i < 0$ for every $1 \leq i \leq r$, and by fairly simple linear algebra we have $a_i \leq 0$ for all i , as noted in [Lip69, page 138, (ii)]. Consider the invertible sheaf $\mathcal{L} = \mathcal{O}_{g^{-1}(\mathcal{U})}(E) \subset \mathcal{O}_{g^{-1}(\mathcal{U})}$ on $g^{-1}(\mathcal{U})$ associated to E , and denote by \mathcal{L}_0 the base change of \mathcal{L} to $g^{-1}(\mathcal{U}) \times_{\mathcal{X}} Z$. Let y be a point of $\widetilde{\mathcal{X}}_0$. If $y \neq x$ then $g_0^{-1}(y)$ is a point and $\mathcal{L}_0|_{g_0^{-1}(y)}$ is therefore ample. On the other

hand, if $y = x$ then $\mathcal{L}_0|_{g_0^{-1}(y)}$ is ample by Kleiman's criterion [Kle66, §III.1]. Since g_0 is proper, by [Gro61, 4.7.1] this implies that the invertible sheaf \mathcal{L}_0 is relatively ample with respect to $g|_{g^{-1}(\mathcal{U})}$, and therefore it is ample since \mathcal{U} is affine. Since \mathcal{U} is algebraized by an open subscheme U of $\widetilde{\mathrm{Bl}_Z X}$, Grothendieck's existence theorem [Gro61, 5.4.5] guarantees that $g^{-1}(\mathcal{U})$ is algebraized by a proper U -scheme. Since those algebraizations are unique, we can glue them and so we deduce that \mathcal{Y} is algebraized by a k -scheme Y , endowed with a proper morphism $g : Y \rightarrow \mathrm{Bl}_Z X$. Set $D = Y \times_X Z$; then D is Cartier in Y by the universal property of $\mathrm{Bl}_Z X$. Since $f : \mathcal{Y} \rightarrow \mathcal{X}$ induces an isomorphism at the level of normalized spaces, g induces an isomorphism out of D , and so (Y, D) is a log modification of (X, Z) algebraizing \mathcal{Y} . \square

(7.7) If $f : \mathcal{Y} \rightarrow \mathcal{X}$ and $f' : \mathcal{Y}' \rightarrow \mathcal{X}$ are two formal log modifications of (X, Z) , we say that \mathcal{Y}' *dominates* \mathcal{Y} if there is a morphism of formal schemes $g : \mathcal{Y}' \rightarrow \mathcal{Y}$ such that $f \circ g = f'$; we denote this by $\mathcal{Y}' \geq \mathcal{Y}$. If such a morphism g exists, then it is unique. Two formal log modifications \mathcal{Y}' and \mathcal{Y} are *isomorphic* if $\mathcal{Y} \geq \mathcal{Y}' \geq \mathcal{Y}$, i.e. if there is an isomorphism $\mathcal{Y}' \cong \mathcal{Y}$ commuting with the morphisms to \mathcal{X} . The domination relation is a filtered partial order on the set of isomorphism classes of formal log modifications of X . By the universal properties of blowup and normalization, this partially ordered set has as unique minimal element \mathcal{X} .

(7.8) If \mathcal{Y} is a formal log modification of (X, Z) , we denote by $\mathrm{Div}_{X,Z}(\mathcal{Y})$ the finite non-empty subset of $T_{X,Z}$ consisting of the $\mathbb{R}_{>0}$ -orbits of the divisorial valuations associated to the components of $\mathcal{Y} \times_{\mathcal{X}} Z$. If \mathcal{Y} is algebraized by a log modification (Y, D) , we will also denote $\mathrm{Div}_{X,Z}(\mathcal{Y})$ by $\mathrm{Div}_{X,Z}(Y)$. We write $\mathrm{Div}_{X,Z}$ for the union of the sets $\mathrm{Div}_{X,Z}(\mathcal{Y})$, for \mathcal{Y} ranging over all the formal log modifications of (X, Z) ; it is the set of the $\mathbb{R}_{>0}$ -orbits of the divisorial valuations on X whose centers lie in Z . We call the elements of $\mathrm{Div}_{X,Z}$ the *divisorial points* of $T_{X,Z}$. We call the finite set $\mathrm{Div}_{X,Z}(\widetilde{\mathcal{X}})$ of divisorial points of $T_{X,Z}$ the *analytic boundary* of $T_{X,Z}$, and we denote it by $\partial^{an}T_{X,Z}$. Since any formal log modification \mathcal{Y} of (X, Z) dominates \mathcal{X} , we always have $\partial^{an}T_{X,Z} \subset \mathrm{Div}_{X,Z}(\mathcal{Y})$.

(7.9) Example. If $X = \mathbb{A}_{\mathbb{C}}^2$ and $Z = \{0\}$, so that $T_{X,Z}$ is the valuative tree as in 3.13, then $\partial^{an}T_{X,Z}$ consists of one point, the $\mathbb{R}_{>0}$ -orbit of the order of vanishing at the origin of the complex plane. This is the point that Favre and Jonsson call the *multiplicity valuation*.

(7.10) Lemma. Let \mathcal{Y} be a formal log modification of (X, Z) , and let x be a closed point of $\mathcal{Y} \times_{\mathcal{X}} Z$. Then:

- (i) $\mathrm{Div}_{X,Z}(\mathcal{Y})$ coincides with the inverse image via the specialization morphism $\mathrm{sp}_{\mathcal{Y}} : T_{X,Z} \rightarrow \mathcal{Y}$ of the set of generic points of the irreducible components of $\mathcal{Y} \times_{\mathcal{X}} Z$;
- (ii) the open subspace $\mathrm{sp}_{\mathcal{Y}}^{-1}(x)$ of $T_{X,Z}$ can be given the structure of a pseudo-affinoid space.

Proof. If η is the generic point of an irreducible component of $\mathcal{Y} \times_{\mathcal{X}} Z$, then the associated divisorial point specializes to η . Being normal, \mathcal{Y} is regular in codimension one, so the ring $\mathcal{O}_{\mathcal{Y},\eta}$ is a regular local ring of dimension one, hence a discrete valuation ring. Therefore, it is the only valuation ring dominating $\mathcal{O}_{\mathcal{Y},\eta}$, which means that there is only one point of $T_{X,Z}$ specializing to η , proving (i). To show (ii), set $\mathcal{U} = \mathrm{Spf}(\widehat{\mathcal{O}_{\mathcal{Y},x}})$. By 2.19, the inverse image of x in $\mathcal{Y}^{\triangleright}$ via the specialization morphism is isomorphic to $\mathcal{U}^{\triangleright}$. Let f_x be a local equation for $\mathcal{Y} \times_{\mathcal{X}} Z$ at x ; we then have $\mathcal{F}_x \cong (\mathcal{U}^{\triangleright} \setminus V(f_x))/\mathbb{R}_{>0} \cong \mathcal{U}_{f_x}^{\triangleright}$, and the latter is pseudo-affinoid. \square

(7.11) We define a family \mathcal{W} of subspaces of $T_{X,Z}$ as follows. Denote by $D = \widetilde{\mathrm{Bl}_Z X} \times_X Z$ the exceptional divisor in $\widetilde{\mathrm{Bl}_Z X}$. A nonempty subspace of $T_{X,Z}$ belongs to \mathcal{W} if and only if it is of the form $\mathrm{sp}_{\mathcal{X}}^{-1}(D \cap U)$, for some affine open U of $\widetilde{\mathrm{Bl}_Z X}$ such that $D \cap U$ is a principal divisor of U .

(7.12) Lemma. *The family \mathcal{W} is an atlas of $T_{X,Z}$.*

Proof. Let $W = \mathrm{sp}_{\mathcal{X}}^{-1}(D \cap U)$ be an element of \mathcal{W} corresponding to some affine open U of $\widetilde{\mathrm{Bl}_Z X}$. Then $W = \mathcal{U}^{\triangleright}$, where \mathcal{U} is the formal completion of U along $D \cap U$. The formal scheme \mathcal{U} is affine since U is affine, and it is of finite type over $k[[t]]$, where the $k[[t]]$ -structure is defined by sending t to an equation for $D \cap U$. Moreover, since $\mathrm{sp}_{\mathcal{X}}^{-1}(D) = T_{X,Z}$ and D is Cartier in $\widetilde{\mathcal{X}}$, the elements of \mathcal{W} cover $T_{X,Z}$ and therefore \mathcal{W} is an atlas of $T_{X,Z}$. \square

(7.13) We call \mathcal{W} the *canonical atlas* of $T_{X,Z}$. The reason this is relevant is the following. If $V \subset T_{X,Z}$ is a $k((t))$ -analytic space which is a union of elements of \mathcal{W} , then the family of analytic subspaces $\mathcal{W}|_V = \{W \in \mathcal{W} \mid W \subset V\}$ is a distinguished formal atlas of V (in the sense of [BL85, §1]) by strict affinoid domains. This means in particular that a formal model of V can be reconstructed by gluing the affine formal $k[[t]]$ -schemes of finite type $\mathrm{Spf}(\mathcal{O}_W^\circ(W))$, for W in $\mathcal{W}|_V$. Moreover, in our situation we can forget the $k[[t]]$ -structures and glue all the affine special formal k -schemes $\mathrm{Spf}(\mathcal{O}_W^\circ(W))$, retrieving the special formal k -scheme $\widetilde{\mathcal{X}}$. A similar idea will be used to construct formal log modifications in Theorem 7.18. Note that the union of the Shilov boundary points of elements of \mathcal{W} is $\partial^{\mathrm{an}} T_{X,Z}$ (see [Ber90, §2.4] for the definition of the Shilov boundary of an affinoid space). In particular, if $V \subset T_{X,Z}$ is a $k((t))$ -analytic space which is a union of elements of \mathcal{W} , then $V \cap \partial^{\mathrm{an}} T_{X,Z}$ is the analytic boundary of V in the sense of [Ber90, §3.1].

(7.14) Lemma. *If \mathcal{Y} is a formal log modification of (X, Z) , then the set of the connected components of $T_{X,Z} \setminus \mathrm{Div}_{X,Z}(\mathcal{Y})$ coincides with the family*

$$\{\mathrm{sp}_{\mathcal{Y}}^{-1}(x) \mid x \in \mathcal{Y} \times_{\mathcal{X}} Z \text{ closed point}\}.$$

Proof. Lemma 7.10 implies that $T_{X,Z} \setminus \mathrm{Div}_{X,Z}(\mathcal{Y}) = \bigcup_x \mathrm{sp}_{\mathcal{Y}}^{-1}(x)$, where this union, taken over the closed points of $\mathcal{Y} \times_{\mathcal{X}} Z$, is disjoint. Each $\mathrm{sp}_{\mathcal{Y}}^{-1}(x)$ is

open by anticointinuity of $\mathrm{sp}_{\mathcal{Y}}$, so it is a union of connected components of $T_{X,Z} \setminus \mathrm{Div}_{X,Z}(\mathcal{Y})$. The fact that $\mathrm{sp}_{\mathcal{Y}}^{-1}(x)$ is connected is [Bos77, 5.9] applied to any $k((t))$ -analytic curve $W \in \mathcal{W}$ such that $x \in \mathrm{sp}_{\mathcal{X}}(W)$. \square

(7.15) If C is a $k((t))$ -analytic curve, its points can be divided into four types, according to the valuative invariants of their completed residue field (see e.g. [Duc, 3.3.2]). In particular a point x of C is said to be of *type 2* if $\mathrm{trdeg}_k \widehat{\mathcal{H}}(x) = 1$, where $\mathcal{H}(x)$ denotes the residue field of $\mathcal{X}(x)$. Assuming k infinite, the points of type 2 are precisely the branching points of C , i.e. a point x of C is of type 2 if and only if $C \setminus \{x\}$ has at least three connected components (and then it has infinitely many). We refer to [Tem10, §6] or [BPR14] for a description of the structure of non-archimedean analytic curves.

(7.16) Lemma. *If x is a point of $T_{X,Z}$, V is a subspace of $T_{X,Z}$ which contains x , and C is a $k((t))$ -analytic curve such that $\mathrm{for}(C) \cong V$, then x is a divisorial point of $T_{X,Z}$ if and only if it is a point of type 2 of C .*

Proof. By abuse of notation we denote by x also a point of \mathcal{X}^* whose image in $T_{X,Z}$ is the given point x . The completed residue field $\mathcal{H}(x)$ of \mathcal{X}^* at x can be computed also as the completed residue field of C at x . Therefore, we deduce that it is a valued extension of $k((t))$ (for some nontrivial t -adic absolute value that we don't need to specify), and in particular

$$\mathrm{rank}_{\mathbb{Q}} |\mathcal{H}(x)^{\times}| / |k^{\times}| \otimes_Z \mathbb{Q} \geq \mathrm{rank}_{\mathbb{Q}} |k((t))^{\times}| / |k((t))^{\times}| \otimes_Z \mathbb{Q} + 1 \geq 1.$$

Moreover, by Abhyankar's inequality (see [Vaq00, Corollaire to 5.5]) we have

$$\mathrm{rank}_{\mathbb{Q}} |\mathcal{H}(x)^{\times}| / |k^{\times}| \otimes_Z \mathbb{Q} + \mathrm{trdeg}_k \widehat{\mathcal{H}}(x) \leq 2.$$

We said that x is a type 2 point of C if and only if $\mathrm{trdeg}_k \widehat{\mathcal{H}}(x) = 1$, and by the two inequalities above this is equivalent to

$$\begin{cases} \mathrm{rank}_{\mathbb{Q}} |\mathcal{H}(x)^{\times}| / |k^{\times}| \otimes_Z \mathbb{Q} = 1 \\ \mathrm{trdeg}_k \widehat{\mathcal{H}}(x) = 1 \end{cases}$$

By [Vaq00, Example 7, Proposition 10.1], this is equivalent to x being a divisorial point of $T_{X,Z}$. \square

(7.17) A *vertex set* of $T_{X,Z}$ is any finite subset of $\mathrm{Div}_{X,Z}$ containing $\partial^{an} T_{X,Z}$.

The following theorem is the main result of this section.

(7.18) Theorem. *Let (X, Z) be as in 7.1. Then the map $\mathcal{Y} \mapsto \mathrm{Div}_{X,Z}(\mathcal{Y})$ is an isomorphism between the following partially ordered sets:*

- (i) *isomorphism classes of formal log modifications of (X, Z) , ordered by domination;*
- (ii) *vertex sets of $T_{X,Z}$, ordered by inclusion.*

Proof. We follow the lines of [Duc, 6.3.15], but the general ideas (over an algebraically closed field) go back to [BL85] and can be found also elsewhere, for example in [BPR14, §4]. The proof will be divided in four steps.

Step 1: Construction of the formal scheme \mathcal{Y} . Let S be a subset of $\text{Div}_{X,Z}$ as in (ii). Let \mathcal{V} be the family of subspaces of $T_{X,Z}$ defined as follows. A compact subspace V of $T_{X,Z}$ belongs to \mathcal{V} if and only if there exist a subset S' of S and a family $\{U_i\}$ of connected components of $T_{X,Z} \setminus S'$ such that the following conditions are satisfied:

- (i) $V \subset W$ for some element W of the canonical atlas \mathcal{W} of $T_{X,Z}$;
- (ii) $V = T_{X,Z} \setminus \coprod U_i$;
- (iii) $V \cap S = S'$;
- (iv) for every $x \in S'$, the set of indices i such that x belongs to the topological boundary $\partial U_i := \overline{U_i} \setminus U_i$ of U_i is finite, and nonempty if $x \notin \partial^{\text{an}} T_{X,Z}$;
- (v) every connected component of $T_{X,Z}$ which does not meet S' is contained in $\coprod U_i$.

Note that the condition (v) is trivially satisfied if Z is connected, since in this case $T_{X,Z}$ is itself connected. By taking $S' = \partial^{\text{an}} T_{X,Z}$ and by choosing an appropriate family $\{U_i\}$ using Lemma 7.14, we can obtain every element of the canonical atlas \mathcal{W} of $T_{X,Z}$ as an element of \mathcal{V} . In particular, the elements of \mathcal{V} cover $T_{X,Z}$. Moreover, observe that \mathcal{V} is closed under intersection, since if V_1 and V_2 are elements of \mathcal{V} corresponding respectively to families $\{U_{1,i}\}$ and $\{U_{2,j}\}$ of subspaces of $T_{X,Z}$, then $V_1 \cap V_2$ is the element of \mathcal{V} corresponding to the family of the connected components of $T_{X,Z} \setminus (S \cap V_1 \cap V_2)$ which are unions of sets of the form $U_{1,i}$ or $U_{2,j}$. Let now W be an element of \mathcal{W} , seen as a $k((t))$ -analytic space and consider the family $\mathcal{V}|_W = \{V \cap W \mid V \in \mathcal{V}\}$ of subspaces of W , seen as $k((t))$ -analytic spaces themselves. Then $\mathcal{V}|_W$ is the same family considered in [Duc, 6.3.15.2], so it is a strict formal affinoid atlas of W with vertex set $S \cap W$. The associated formal $k[[t]]$ -scheme \mathcal{Y}_W is therefore a formal model of W with vertex set $S \cap W$. Recall that as noted in 5.5 the canonical model of an affinoid domain of $T_{X,Z}$ doesn't depend on a $k((t))$ -analytic structure. This guarantees that we can glue all the \mathcal{Y}_W , seen as affine special formal k -schemes, to a special formal k -scheme \mathcal{Y} .

Step 2: \mathcal{Y} is a formal log modification of (X, Z) . The formal scheme \mathcal{Y} is defined by gluing affine special formal k -schemes of the form $\text{Spf}(\mathcal{O}_{T_{X,Z}}^\circ(V))$, for V ranging among the elements of \mathcal{V} . By [dJ95, 7.4.2] (as recalled in 4.13) each $\mathcal{O}_{T_{X,Z}}^\circ(V)$ is integrally closed in its ring of fractions, and therefore \mathcal{Y} is normal. For each $W \in \mathcal{W}$, the inclusions $V \rightarrow W$, for $V \in \mathcal{V}|_W$, induce an adic morphism $\mathcal{Y}_W \rightarrow \widetilde{\mathcal{X}}$. These morphisms glue to an adic morphism $\mathcal{Y} \rightarrow \widetilde{\mathcal{X}}$, so we obtain an adic morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ and $\mathcal{Y} \times_{\mathcal{X}} Z$ is Cartier in \mathcal{Y} . Moreover, since $T_{\mathcal{Y}_W} = W$ for every W in the covering \mathcal{W} , \mathcal{Y} is a model of $T_{X,Z}$. Therefore, \mathcal{Y} is a formal log modification of (X, Z) .

Step 3: Bijectivity of the correspondence. By construction, each element of \mathcal{V} is a strict $k((t))$ -affinoid, and it is the inverse image through the specialization morphism of an affine open subspace of \mathcal{Y} in which $\mathcal{Y} \times_{\mathcal{X}} Z$ is principal, so it follows from [Ber90, 2.4.4] that we have $\text{Div}_{X,Z}(\mathcal{Y}) = S$. This

shows that the map $\mathcal{Y} \mapsto \operatorname{Div}_{X,Z}(\mathcal{Y})$ is surjective. To prove its bijectivity, we need to show that a formal log modification of (X, Z) is determined by its divisorial set. This can be done locally, again using [Duc, 6.3.15], as follows. Assume that \mathcal{Y}' is another formal log modification of (X, Z) such that $\operatorname{Div}_{X,Z}(\mathcal{Y}') = S$. Let W be the element of the canonical atlas \mathcal{W} associated to an open affine subspace U of \tilde{X} . Then again by [Duc, 6.3.15] the $k[[t]]$ -subspace $\tau^{-1}(U)$ of \mathcal{Y}' , where we denote by τ the composition of the canonical map $Y' \rightarrow \tilde{X}$ with $\operatorname{sp}_{\mathcal{Y}'}$, is isomorphic to the open \mathcal{Y}_W of \mathcal{Y} , and hence \mathcal{Y}' is isomorphic to \mathcal{Y} .

Step 4: Functoriality. It is clear that if \mathcal{Y} and \mathcal{Y}' are two formal log modifications of (X, Z) such that \mathcal{Y}' dominates \mathcal{Y} , then $\operatorname{Div}_{X,Z}(\mathcal{Y}) \subset \operatorname{Div}_{X,Z}(\mathcal{Y}')$. To show that the bijective correspondence that we have constructed respects the partial orders it is then enough to note the following. Let $S_1 \subset S_2$ be finite nonempty subsets of $\operatorname{Div}_{X,Z}$, and let \mathcal{Y}_1 and \mathcal{Y}_2 be the corresponding formal models, defined using formal atlases \mathcal{V}_1 and \mathcal{V}_2 . Then from the definition of the atlases \mathcal{V}_i it follows that we can cover $T_{X,Z}$ by $V_{1,1}, \dots, V_{1,r} \in \mathcal{V}_1$ and also by $V_{2,1}, \dots, V_{2,r} \in \mathcal{V}_2$ in such a way that $V_{2,i}$ is a subspace of $V_{1,i}$ for every i . This gives a morphism $\mathcal{Y}_2 \rightarrow \mathcal{Y}_1$ commuting with the two morphisms $\mathcal{Y}_i \rightarrow \mathcal{X}$, hence a morphism of formal log modifications.

This completes the proof of Theorem 7.18. \square

(7.19) Remarks. If X has only rational singularities or if k is the algebraic closure of the field \mathbb{F}_p for some prime number p , then Theorem 7.18 can also be proved using resolution of singularities to find a suitable log modification and then contractibility results [Art62, 2.3, 2.9] to contract all unnecessary divisors, and every formal log modification of (X, Z) is algebraizable. In general not all of the formal log modifications given by Theorem 7.18 are algebraizable, but the contractibility criterion of Grauert-Artin [Art70] guarantees that they can always be algebraized in the category of algebraic spaces over k . Moreover, since Artin proved that a smooth algebraic space in dimension 2 is a scheme, we retrieve Proposition 7.6.

8. DISCS AND ANNULI

In this section we will study one-dimensional open discs and open annuli in normalized spaces. The main result, Proposition 8.8, explains in which sense those discs and annuli are determined by their canonical reduction.

(8.1) We say that a $k((t))$ -analytic space X is *pseudo-affinoid* if it is the Berkovich space of an affine special formal $k[[t]]$ -scheme. When this is the case, [dJ95, 7.4.2] (recalled in 4.13) tells us that $X \cong \mathcal{X}_t^\natural$, where $\mathcal{X}_t = \operatorname{Spf}(\mathcal{O}_X^\circ(X))$ is called the *canonical formal model* of X , and moreover \mathcal{X}_t is integrally closed in its generic fiber. We call *canonical reduction* of X the reduced affine special formal k -scheme $((\mathcal{X}_t)_s)_{\operatorname{red}}$ associated to the special fiber $(\mathcal{X}_t)_s = \mathcal{X}_t \otimes_{k[[t]]} k$ of \mathcal{X}_t , and we denote it by X_0 .

As for affinoids, we say that a pseudo-affinoid $k((t))$ -analytic space X is

principal if the special fiber $(\mathcal{X}_t)_s$ of its canonical formal model is already reduced, i.e. if it coincides with X_0 .

(8.2) We say that a normalized k -space Y is *pseudo-affinoid* (respectively, *principal pseudo-affinoid*) if it is of the form $\text{for}(X)$, with X a pseudo-affinoid (resp. a principal pseudo-affinoid) $k((t))$ -analytic space. This coincides with the definition in 5.11. The affine special formal k -scheme $\mathcal{Y} = \text{Spf}(\mathcal{O}_Y^\circ(Y))$ is called the *canonical formal model* of Y . Note that this is an abuse of notation since \mathcal{Y} is not a formal model of the normalized k -space Y .

We define the *canonical reduction* of Y as the closed formal subscheme Y_0 of \mathcal{Y} defined by the ideal $I = \bigcap \sqrt{(f)}$, where the intersection is taken over all elements f of $\mathcal{O}_Y^\circ(Y)$ which do not vanish on Y . Being an intersection of radical ideals, I is radical itself, so Y_0 is a reduced special formal k -scheme. This definition is consistent with the previous one, since Y_0 is isomorphic to the canonical reduction X_0 of X . To prove this, we need to show that $I = \sqrt{(t)}$. Clearly $I \subset \sqrt{(t)}$ since t does not vanish on Y . By Theorem 4.9 we have $Y = \text{for}(X) = T_{\mathcal{X}} \setminus V(t)$, where \mathcal{X}_t is the canonical model of X and by abuse of notation we denoted by t the image of t in $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Therefore, if f doesn't vanish on Y we must have $V(f) \subset V(t)$, and so $\sqrt{(t)} \subset \sqrt{(f)}$, which implies that $\sqrt{(t)} \subset I$. Moreover, remark that Y is principal if and only if the ideal I defined above is a principal ideal. Mind that whether a pseudo-affinoid $k((t))$ -analytic space X is principal is not determined by the normalized space $\text{for}(X)$, as the example in 5.5 shows.

(8.3) A $k((t))$ -analytic space is called an *open $k((t))$ -disc*, or simply a *disc*, if it is isomorphic to $\text{Spf}(k[[t]][[X]])_t^\triangleright$.

Equivalently, a disc is a $k((t))$ -analytic space isomorphic to the subspace of $\mathbb{A}_{k((t))}^{1,an}$ defined by the inequality $|T| < 1$, where $\mathbb{A}_{k((t))}^1 = \text{Spec}(k((t))[T])$.

(8.4) A $k((t))$ -analytic space is called an *open $k((t))$ -annulus of modulus n* , or simply an *annulus of modulus n* , if it is of the form $A_n := \text{Spf}(k[[t]][[X, Y]]/(XY - t^n))_t^\triangleright$, for some $n > 0$.

Equivalently, an annulus of modulus n is a $k((t))$ -analytic space isomorphic to the subspace of $\mathbb{A}_{k((t))}^{1,an}$ defined by the inequality $|t^n| < |T| < 1$.

We call *standard annulus* an annulus of modulus one. Remark that an annulus is standard if and only if it has no $k((t))$ -point.

(8.5) The modulus of an annulus X is well defined, and depends only on the algebra $\mathcal{O}_X^\circ(X)$, hence only on the normalized space $\text{for}(X)$. Indeed, if we denote by $\mathfrak{M} = (t, X, Y)$ the maximal ideal of $\mathcal{O}_X^\circ(X)$, then the modulus of X is the smallest $i > 0$ such that the inequality $\dim_k \frac{\mathfrak{M}^i}{\mathfrak{M}^{i+1}} < \binom{i+2}{2}$ holds.

(8.6) Discs and annuli are principal pseudo-affinoid $k((t))$ -analytic spaces. Indeed, the canonical formal model of a disc is the affine formal $k[[t]]$ -scheme $\text{Spf}(k[[t]][[X]])_t$, whose special fiber is $\text{Spf}(k[[X]])$; while the canonical formal model of an annulus of modulus n is the affine formal $k[[t]]$ -scheme $\text{Spf}(k[[t]][[X, Y]]/(XY - t^n))_t$, whose special fiber is $\text{Spf}(k[[X, Y]]/(XY))$.

(8.7) Remark. The canonical model of an annulus is regular if and only if the annulus is standard. Indeed, the maximal ideal of $k[[t, X, Y]]/(XY - t^n)$ is $\mathfrak{M} = (t, X, Y)$, hence $\mathfrak{M}^2 = (t^2, X^2, Y^2, tX, tY, t^n)$ so the k -vector space $\mathfrak{M}/\mathfrak{M}^2$ has dimension 2, with basis $\{X, Y\}$, if and only if $n = 1$.

It is clear that any two $k((t))$ -discs are always isomorphic as $k((t))$ -analytic spaces, and that two $k((t))$ -annuli are isomorphic if and only if they have the same modulus. In our setting we need something stronger, which will be the content of the next proposition and of the corollary following it.

(8.8) Proposition. *Let X be a principal pseudo-affinoid $k((t))$ -analytic space, and denote by X_0 its canonical reduction. Then:*

- (i) *if $X_0 \cong \mathrm{Spf}(k[[X]])$, X is a $k((t))$ -disc;*
- (ii) *if $X_0 \cong \mathrm{Spf}(k[[X, Y]]/(XY))$ and X is irreducible, then X is a $k((t))$ -annulus.*

Proof. Part (i) follows easily from the uniqueness of deformations of smooth affine formal schemes, see [PR08]. Indeed, up to isomorphism there is only one affine special formal $k[[t]]$ -scheme whose special fiber is $\mathrm{Spf}(k[[X]])$, so the canonical formal model of X is isomorphic to $\mathrm{Spf}(k[[t]][[X]])$, hence X is a $k((t))$ -disc. To prove (ii) we make use of the fact that the miniversal deformation of the formal node $\mathrm{Spf}(k[[X, Y]]/(XY))$ is $\mathrm{Spf}(k[[t, X, Y]]/(XY - t))$, which is [Har10, 14.0.1]. This means that if $\mathcal{X} = \mathrm{Spf}(A)$ is an affine and flat special formal $k[[t]]$ -scheme whose special fiber is isomorphic to $\mathrm{Spf}(k[[X, Y]]/(XY))$, then there is a local k -algebra morphism $\varphi: k[[t]] \rightarrow k[[t']]$ such that $\mathcal{X} \cong \mathrm{Spf}(k[[t, X, Y]]/(XY - t)) \otimes_{k[[t]]} k[[t']]$. The morphism φ is determined by a power series $\varphi(t) = T(t') \in k[[t']]$ such that $T(0) = 0$, so we have $\mathcal{X} \cong \mathrm{Spf}(k[[t', X, Y]]/(XY - T(t')))$. Moreover, since X is irreducible then $T(t')$ cannot be zero. The power series $T(t')$ can be written as $u(t')^n$ for some unit u of $k[[t']]$, hence by further sending X to uX we obtain an isomorphism $\mathcal{X} \cong \mathrm{Spf}(k[[t', X, Y]]/(XY - (t')^n))$. By applying this to the canonical $k[[t]]$ -formal model of X , we deduce that X is a $k((t))$ -annulus. \square

(8.9) Corollary. *Let X be a $k((t))$ -disc or a $k((t))$ -annulus and let Y be a principal pseudo-affinoid $k((t))$ -analytic space such that $\mathrm{for}(X) \cong \mathrm{for}(Y)$. Then X and Y are isomorphic as $k((t))$ -analytic spaces.*

Proof. This follows from Proposition 8.8 since the special fiber of the canonical formal model of a principal pseudo-affinoid $k((t))$ -analytic space does not depend on the chosen principal pseudo-affinoid $k((t))$ -analytic structure. For annuli, note that the modulus of Y is the same as the modulus of X by 8.5. \square

(8.10) The last result allows us to define more intrinsically discs and annuli in normalized spaces: we say that a principal pseudo-affinoid analytic domain V of a normalized k -space T is a *disc* (respectively an *annulus of modulus n*) if there exists a $k((t))$ -analytic space X such that $V \cong \mathrm{for}(X)$ and X is a disc (resp. an annulus of modulus n). Corollary 8.9 tells us that this property is independent on the choice of a principal pseudo-affinoid structure on V .

9. FORMAL FIBERS

In this section we move to the study of the fibers of the specialization morphism. For normal surfaces we will get in Proposition 9.5 very explicit results involving discs and annuli, analogous to [BL85, 2.2 and 2.3]. For simplicity, from now on we assume that k is algebraically closed.

(9.1) Let \mathcal{X} be a special formal k -scheme and let x be a point of \mathcal{X} . We call *formal fiber* of x the inverse image $\mathcal{F}_x := \mathrm{sp}_{\mathcal{X}}^{-1}(x)$ of x in $T_{\mathcal{X}}$ via the specialization morphism. It is a subspace of $T_{\mathcal{X}}$, open if x is closed in \mathcal{X} .

(9.2) Let X be a normal scheme of finite type over k , Z a divisor of X and $\mathcal{X} = \widehat{X/Z}$ the formal completion of X along Z . Then the argument of Lemma 7.10 tells us that, if η is a generic point of an irreducible component of Z , \mathcal{F}_{η} is a single point of $T_{\mathcal{X}}$, precisely the point corresponding to the $\mathbb{R}_{>0}$ -orbit of divisorial valuations associated to the component $\{\eta\}$. If x is a closed point of \mathcal{X} , \mathcal{F}_x can be given the structure of a pseudo-affinoid space.

(9.3) Lemma. *Let \mathcal{X} be a normal special formal k -scheme of dimension n and let x be a closed point of \mathcal{X} such that there exists an ideal of definition of \mathcal{X} which is principal at x . Then \mathcal{X} is regular at x if and only if $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}(\mathcal{F}_x) \cong k[[X_1, \dots, X_n]]$.*

Proof. Consider the normal special formal k -scheme $\mathcal{U} = \mathrm{Spf}(\widehat{\mathcal{O}_{\mathcal{X},x}})$. By 2.19, the inverse image of x in $\mathcal{X}^{\triangleright}$ via the specialization morphism is isomorphic to $\mathcal{U}^{\triangleright}$, so \mathcal{F}_x is isomorphic to $(\mathcal{U}^{\triangleright} \setminus V(t))/\mathbb{R}_{>0} \cong \mathcal{U}_t^{\triangleright}$, where we have denoted by t a local generator of an ideal of definition of \mathcal{X} at x . It follows that $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}(\mathcal{F}_x) \cong \mathcal{O}_{\mathcal{U}_t^{\triangleright}}^{\circ}$, and since \mathcal{U}_t is normal this is also equal to $\mathcal{O}_{\mathcal{U}_t}(\mathcal{U}_t) \cong \widehat{\mathcal{O}_{\mathcal{X},x}}$ by [dJ95, 7.3.6]. To conclude, by Cohen's structure theorem [Coh46] x is regular in \mathcal{X} if and only if $\widehat{\mathcal{O}_{\mathcal{X},x}} \cong k[[X_1, \dots, X_n]]$. \square

(9.4) Remark. Note that in the lemma above, while the k -algebra $\mathcal{O}_{T_{\mathcal{X}}}^{\circ}(\mathcal{F}_x)$ doesn't depend on the geometry of \mathcal{X}_0 around x , its largest ideal of definition, and therefore also the space \mathcal{F}_x , strongly depends on it. Focusing now on the case of surfaces, an example of this behavior is detailed in the following proposition, which is the analogue for normalized spaces of surfaces of a classical result of Bosch and Lütkebohmert [BL85, 2.2 and 2.3] (see also [Ber90, 4.3.1] for a formulation in the language of Berkovich curves).

(9.5) Proposition. *Let \mathcal{X} be a normal special formal k -scheme of dimension 2 and let x be a closed point of \mathcal{X} such that there exists an ideal of definition of \mathcal{X} which is principal at x . Then:*

- (i) x is regular both in \mathcal{X} and in \mathcal{X}_0 if and only if its formal fiber \mathcal{F}_x is a disc;
- (ii) x is regular in \mathcal{X} and an ordinary double point in \mathcal{X}_0 if and only if \mathcal{F}_x is a standard annulus.

Proof. We endow \mathcal{F}_x with the pseudo-affinoid $k((t))$ -analytic structure from Lemma 7.10. Assume that x is regular both in \mathcal{X} and in \mathcal{X}_0 . Then \mathcal{X}_0 is itself principal locally at x , and so we can choose an element t of $\mathcal{O}_{\mathcal{X},x}$ which locally defines \mathcal{X}_0 . Then the image \bar{t} of t in $\mathfrak{M}/\mathfrak{M}^2$ doesn't vanish, where we have denoted by \mathfrak{M} the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. We then pick another element U of \mathfrak{M} whose image in $\mathfrak{M}/\mathfrak{M}^2$ generates it together with \bar{t} . Cohen's Theorem gives us an isomorphism $\widehat{\mathcal{O}_{\mathcal{X},x}} \cong k[[t, U]]$, therefore the formal fiber \mathcal{F}_x is the disc $\mathcal{F}_x = \mathrm{Spf}(k[[t, U]])_t^\natural$. Conversely, if \mathcal{F}_x is a disc, isomorphic to $\mathrm{Spf}(k[[t, U]])_t^\natural$, then $\mathcal{O}_{T_{X,Z}}^\circ(\mathcal{F}_x) \cong k[[t, U]]$, so \mathcal{X} is regular at x by Lemma 9.3, and \mathcal{X}_0 is locally defined by the equation $t = 0$, so it is itself regular, proving (i). The proof of (ii) is similar. If x is an ordinary double point of \mathcal{X}_0 then we can find elements U and V of \mathfrak{M} whose images in $\mathfrak{M}/\mathfrak{M}^2$ generate it and such that \mathcal{X}_0 is defined at x by the equation $UV = 0$. Then we obtain $\mathcal{F}_x = \mathrm{Spf}(k[[U, V]])_{UV}^\natural = \mathrm{Spf}(k[[t]][[U, V]]/(UV - t))_t^\natural$, which is a standard annulus. The converse implication is proved as in part (i). \square

10. LOG ESSENTIAL VALUATIONS

In this section we make use of the previous results to characterize in terms of the structure of $T_{X,Z}$ the finite sets of divisorial points of $T_{X,Z}$ which correspond via Theorem 7.18 to the log resolutions of a pair (X, Z) . We will then be able to describe the divisorial points corresponding to the minimal log resolution of (X, Z) , and deduce a characterization of the log essential valuations of (X, Z) . Note that in this section we use the existence of resolution of singularities for surfaces. More precisely, we admit the fact that given any set S of divisorial points of $T_{X,Z}$ there exists a log resolution Y of (X, Z) such that $S \subset \mathrm{Div}_{X,Z}(Y)$. On the other hand, we do not need to assume the existence of a minimal log resolution of (X, Z) . Indeed, this fact will be a consequence of Theorem 10.4.

(10.1) Let (X, Z) be as in 7.1. We say that a vertex set S of $T_{X,Z}$ is *regular* if the connected components of $T_{X,Z} \setminus S$ are discs and a finite number of standard annuli.

Putting together Lemma 7.14, Proposition 9.5 and Proposition 7.6 we obtain the following important result.

(10.2) Proposition. *Let k be an algebraically closed field and let (X, Z) be as in 7.1. Then a formal log modification \mathcal{Y} of (X, Z) is a log resolution if and only if $\mathrm{Div}_{X,Z}(\mathcal{Y})$ is a regular vertex set of $T_{X,Z}$.*

(10.3) We call *log essential valuations* of the pair (X, Z) the elements of $\mathrm{Div}_{X,Z}(Y_{\min})$, where Y_{\min} is the minimal log resolution of (X, Z) . This is exactly the set of divisorial valuations whose center on every log resolution of (X, Z) is a divisor. We call an open subspace U of $T_{X,Z}$ *simple* if it is isomorphic to either a disc or a standard annulus, $U \cap \partial^{\mathrm{an}} T_{X,Z} = \emptyset$ and the topological boundary $\partial U = \overline{U} \setminus U$ is contained in $\mathrm{Div}_{X,Z}$. Then a finite subset

S of $T_{X,Z}$ is a regular vertex set if and only if all the connected components of $T_{X,Z} \setminus S$ are simple subspaces of $T_{X,Z}$.

(10.4) Theorem. *Let k be an algebraically closed field, let (X, Z) be as in 7.1 and let v be an element of $\text{Div}_{X,Z}$. Then v is a log essential valuation of (X, Z) if and only if it has no simple neighborhood in $T_{X,Z}$.*

Before proving this theorem we need a simple result in intersection theory.

(10.5) Lemma. *Let $\pi : \tilde{X} \rightarrow X$ be a proper birational morphism of smooth k -surfaces. Let D be a prime divisor of X and denote by \tilde{D} its strict transform in \tilde{X} . Then the inequality $\tilde{D}^2 < D^2$ holds.*

Proof. We have $\tilde{D} = \pi^*D - E$, where E is an effective divisor of \tilde{X} not containing \tilde{D} , and so $\tilde{D}^2 = (\pi^*D)^2 + E^2 - 2h^*D \cdot E$. But $(\pi^*D)^2 = D^2$, $E^2 < 0$ since E can be contracted, and $h^*D \cdot E = 0$ by the projection formula [Kol96, VI.2.11] since h_*E is a point, proving what we wanted. \square

Proof of Theorem 10.4. Denote by S the set of divisorial points which have no simple neighborhood in $T_{X,Z}$. As observed in 10.3, S is contained in every regular vertex set of $T_{X,Z}$, so in particular it is a vertex set. We have to show is that S is itself regular, so that the associated formal log modification \mathcal{V} of (X, Z) is a log resolution. Choose a regular vertex set S_1 of $T_{X,Z}$, corresponding to a formal log resolution \mathcal{V}_1 of (X, Z) ; then S_1 contains S . We can assume that $S_1 \setminus S$ is nonempty or there is nothing to prove, so we can find a simple subspace of $T_{X,Z}$ containing a point of S_1 . By the definition of S , we have $V \cap S = \emptyset$. Let now (Y_2, D_2) be a log resolution of (X, Z) such that $S_2 = \text{Div}_{X,Z}(\mathcal{V}_2)$ contains $S_1 \cup \partial V$. Set $S_3 = S_2 \setminus V$ and let \mathcal{V}_3 be the associated formal log modification of (X, Z) . The connected components of $T_{X,Z} \setminus S_3$ are either V or connected components of $T_{X,Z} \setminus S_2$, so S_3 is a regular vertex set. Therefore, \mathcal{V}_3 is algebraized by a log resolution (Y_3, D_3) of (X, Z) . Since $S_3 \subset S_2$, we have a corresponding proper birational morphism of smooth varieties $Y_2 \rightarrow Y_3$, which is a composition of blowups in closed points by Zariski's theorem [Har77, V.5.4]. Therefore, there are a point x of $V \cap S_1$ and a log resolution Y_4 in this composition of blowups such that $V \cap S_1 \subset S_4 = \text{Div}_{X,Z}(Y_4)$ and $D_4 = \overline{\{\text{sp}_{Y_4}(x)\}}$ is a prime divisor of Y_4 of self-intersection number -1 . Note that since $S_1 \setminus V \subset S_2 \setminus V = S_3 \subset S_4$ we have $S_1 \subset S_3 \cup \{x\}$ and so $S_1 \subset S_4$, therefore there is a proper birational morphism of smooth varieties $Y_4 \rightarrow Y_1$. Lemma 10.5 applied to this morphism and to the divisors D_4 and $D_1 = \overline{\{\text{sp}_{Y_1}(x)\}}$ tells us that the latter has self-intersection number at least -1 . But since D_1 is contracted by the morphism $Y_1 \rightarrow Y$, we must have $(D_1)^2 < 0$ and therefore $(D_1)^2 = -1$. Finally, Castelnuovo's contractibility criterion [Har77, V.5.7] tells us that D_1 can be contracted to a smooth surface Y_5 . This shows that that $S_5 = S_1 \setminus \{x\}$ is again a regular vertex set of $T_{X,Z}$. By repeating the same argument we can remove all the elements of $S' \setminus S$ one by one, proving that the vertex set S is regular. \square

(10.6) Remark. We have seen in 4.12 that there is a point in the valuative tree which is not divisorial and whose complement is a disc U . This shows that it is really necessary to impose the condition on the topological boundary in the definition of simple subspace, and also to add the points of $\partial^{an}T_{X,Z}$ to the characterization of Theorem 10.4, since in this case a simple neighborhood of the boundary point of T is obtained by taking a disc slightly smaller than U . We expect things to behave better if we take for Z the singular locus of X , seen as a subscheme with a suitable, possibly nonreduced, structure.

(10.7) Remark. Let us assume that $k = \mathbb{C}$ and that $Z = X_{sing}$. Then log essential valuations are not the same as the essential valuations studied by Nash in [Nas95]. Indeed, these two classes differ when the minimal resolution of the pair (X, X_{sing}) is not a log resolution, i.e. when its exceptional locus is not a normal crossing divisor. However, essential and log essential valuations coincide for big classes of singularities, for example for rational singularities. Moreover, since the definition of log resolution requires that the schematic inverse image of the singular locus, and not its set-theoretic inverse image, is a divisor with normal crossing, the minimal log resolution of (X, X_{sing}) might strictly dominate the pair (Y, D) which is usually called the minimal good resolution of X . However, Proposition 9.5 depends only on the reduced scheme underlying D , and can therefore be applied verbatim to $\mathcal{Y} = \widehat{Y/D}$. We deduce that every divisorial point of $T_{X,Z}$ which is not in $\text{Div}_{X,Z}(Y)$ has a neighborhood in $T_{X,Z}$ which is a disc or a standard annulus.

(10.8) Remark. We expect the approach used in this chapter to lead to a new proof of the existence of resolutions of surfaces, at least in characteristic 0, in a similar way as one can prove the semistable reduction theorem for curves using non-archimedean analytic spaces. A proof would go roughly as follows. The normalized space $T_{X,Z}$ can be covered by finitely many smooth affinoid $k((t))$ -analytic curves, since all the points of \mathcal{X}^* are regular. Then [Duc, 5.1.14] applied to those $k((t))$ -analytic curves gives us a vertex set $S \subset \text{Div}_{X,Z}$ such that all connected components of $T_{X,Z} \setminus S$ are *virtual discs* or *virtual annuli*, i.e. $k((t))$ -analytic spaces which become a $k((t))$ -disc or a $k((t))$ -annulus after a finite extension of $k((t))$. If we could prove that all those virtual discs and annuli are actual discs and annuli, we would obtain a log resolution of (X, Z) , since by enlarging S we can cut an annulus of modulus n into n standard annuli. If the characteristic of k is zero, by a special case of [Duc13] every virtual disc is a disc. A virtual annulus is a pseudo-affinoid $k((t))$ -analytic space, and to prove that it is an annulus it would be enough, by a slight generalization of Proposition 8.8, to show that it is a principal pseudo-affinoid. We believe that it is always possible to enlarge S further and break a given virtual annulus in discs and finitely many annuli.

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